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- This would give a plane drawing of $K_5$, a contradiction!

Therefore, $\text{cr}(K_6) = 3$. 

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**Fact:** $\theta(K_n) = \begin{cases} \left\lfloor \frac{n + 7}{6} \right\rfloor & n \neq 9, 10 \\ 3 & n = 9, 10 \end{cases}$

Proved by Beineke, Harary, Vasak, Alekseev, Gonchakov
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Embedding on higher genus surfaces changes Euler’s formula!

**Theorem.** Let $G$ be a graph of genus $g$. Suppose you have an embedding of $G$ on a surface of genus $g$ with no crossings. If $r$ is the number of regions, then $p - q + r = 2 - 2g$.

**Example.** In our embedding of $K_5$ on the torus (genus 1):
Planarity statistics for complete graphs:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>$cr(K_n)$</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>18</td>
<td>36</td>
<td>60</td>
<td>100</td>
<td>150</td>
<td>225</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$\theta(K_n)$</td>
<td>1</td>
<td>2</td>
<td>2</td>
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<td>2</td>
<td>3</td>
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<td>0 1 3 9 18 36 60 100 150 225</td>
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<td>( \theta(K_n) )</td>
<td>1 2 2 2 2 3 3 3 3 3 3 3 4 4</td>
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<tr>
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The crossing number of a complete graph is unknown for \( n \geq 13 \).

**Conjecture.** (Guy, 1972) The crossing number of a complete graph is

\[
    cr(G) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor
\]

The cases \( cr(K_{11}) = 100 \) and \( cr(K_{12}) = 150 \) were proved in 2007.