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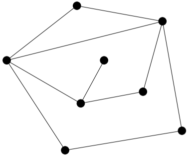
Definition: A **planar graph** is a graph that has a plane drawing.

Example. K_4 is a planar graph because
is a plane drawing of K_4.

Vertices, Edges, and Faces



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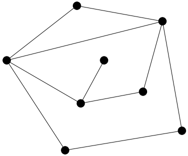

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Notation. Let $p = \#$ of vertices, $q = \#$ of edges, $r = \#$ of regions. Compute the following data:

Graph	p	q	r	
Tetrahedron				
Cube				
Octahedron				
Dodecahedron				
Icosahedron				

In 1750, Euler noticed that ____________ in each of these examples.

Euler's Formula

Theorem 8.1.1 (Euler's Formula) If G is connected, then in a plane drawing of G, $p - q + r = 2$.

Proof (by induction on the number of cycles)
Base Case: If G is a tree, there is one region, so

$$p - q + r = p - (p - 1) + 1 = 2.$$

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Let C be a cycle in G. Let e be any edge in C, then e is adjacent to two different regions, one inside C and one outside C.

$G \setminus e$ has fewer cycles than G, and one fewer region. The inductive hypothesis holds for $G \setminus e$, giving

Maximal Planar Graphs

A graph with "too many" edges isn't planar; how many is too many?

Goal: Find a numerical characterization of "too many"

Definition: A planar graph is called **maximal planar** if adding an edge between any two non-adjacent vertices results in a non-planar graph.

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What do we notice about these graphs?

Numerical Conditions on Planar Graphs

- Every face of a maximal planar graph is a triangle!

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Now substitute into Euler's formula:

Do we need $p \geq 3$?

Numerical Conditions on Planar Graphs

Corollary 8.1.3. Every planar graph with $p \geq 3$ vertices has at most $3p - 6$ edges.

- Start with any planar graph G with p vertices and q edges.
- Add edges to G until it is maximal planar. (with $Q \geq q$ edges.)
- This resulting graph satisfies $Q = 3p - 6$; hence $q \leq 3p - 6$.

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Theorem 8.1.4. The graph K_5 is not planar.

Proof.

Numerical Conditions on Planar Graphs

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Theorem 8.1.5.⋆ If G is planar with girth ≥ 4, then $q \leq 2p - 4$.

Proof. Modify the above proof—instead of $3r = 2q$, we know $4r \leq 2q$. This implies that

$$2 = p - q + r \leq p - q + \frac{2q}{4} = p - \frac{q}{2}.$$

Therefore, $q \leq 2p - 4$.

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Theorem 8.1.5. If G is planar and bipartite, then $q \leq 2p - 4$.

Theorem 8.1.6. $K_{3,3}$ is not planar.

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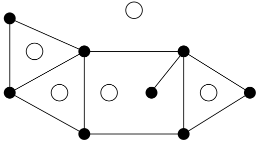
Theorem 8.1.6. $K_{3,3}$ is not planar.

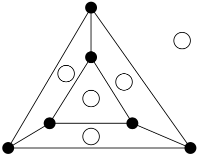
Theorem 8.1.7. Every planar graph has a vertex with degree ≤ 5.

Proof.

Dual Graphs

Definition: Given a plane drawing of a planar graph G, the **dual graph** $D(G)$ of G is a graph with vertices corresponding to the regions of G. Two vertices are connected by an edge each time the two regions share an edge as a border.





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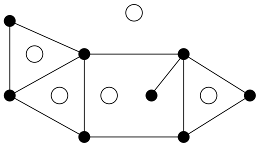

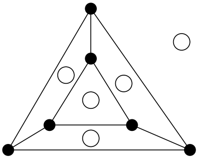

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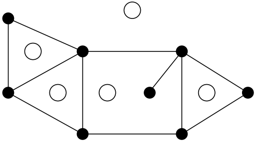

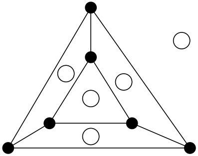

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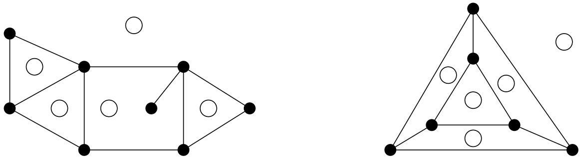

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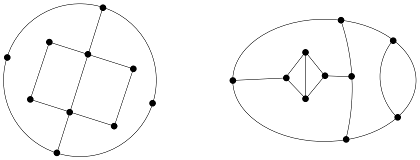
Definition: A graph G is **self-dual** if G is isomorphic to $D(G)$.

Maps

Definition: A *map* is a plane drawing of a connected, bridgeless, planar multigraph. If the map is 3-regular, then it is a **normal map**.



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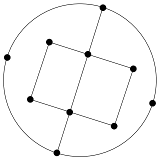
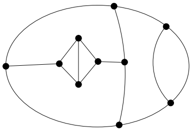




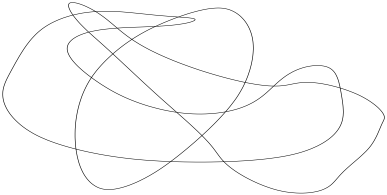
Definition: In a map, the regions are called **countries**. Countries may share several edges.

Definition: A **proper coloring** of a map is an assignment of colors to each country so that no two adjacent countries are the same color.

Question. How many colors are necessary to properly color a map?

Proper Map Colorings

Lemma 8.2.2. If M is a map that is a union of simple closed curves, the regions can be colored by two colors.



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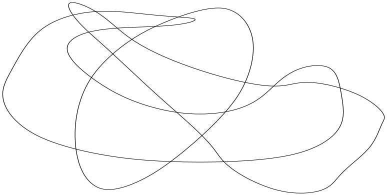

Proof. Color the regions R of M as follows:

$$\left\{\begin{matrix} \text{black} & \text{if } R \text{ is enclosed in an odd number of curves} \\ \text{white} & \text{if } R \text{ is enclosed in an even number of curves} \end{matrix}\right\}.$$

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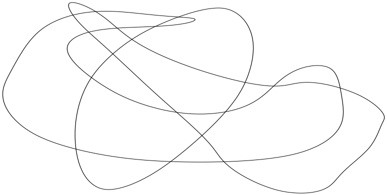

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This is a proper coloring of M. Any two adjacent regions are on opposite sides of a closed curve, so the number of curves in which each is enclosed is off by one.

The Four Color Theorem

Lemma 8.2.6. (The Four Color Theorem)
Every normal map has a proper coloring by four colors.

Proof. Very hard.

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Theorem. If G is a plane drawing of a maximal planar graph, then its dual graph $D(G)$ is a normal map.

- Every face of G is a triangle $\rightsquigarrow$
- G is connected $\rightsquigarrow$
- G is planar $\rightsquigarrow$

The Four Color Theorem

Assuming Lemma 8.2.6,

$$\begin{aligned} G \text{ is maximal planar} &\Rightarrow D(G) \text{ is a normal map} \\ &\Rightarrow \text{countries of } D(G) \text{ 4-colorable} \\ &\Rightarrow \text{vertices of } G \text{ 4-colorable} \\ &\Rightarrow \chi(G) \leq 4 \end{aligned}$$

This proves:

Theorem 8.2.8. If G is maximal planar, then $\chi(G) \leq 4$.

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Since every planar graph is a subgraph of a maximal planar graph, Lemma C implies:

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