Planarity

Up until now, graphs have been completely abstract.
In Topological Graph Theory, it matters how the graphs are drawn.

- Do the edges cross?
- Are there knots in the graph structure?

Definition: A **drawing** of a graph G is a pictorial representation of G in the plane as points and line segments. The line segments must be **simple curves**, which means no intersections are allowed.

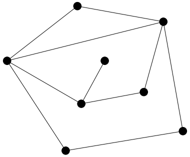
Definition: A **plane drawing** of a graph G is a drawing of the graph in the plane with no crossings. Otherwise, G is **nonplanar**.

Definition: A **planar graph** is a graph that has a plane drawing.

Example. K_4 is a planar graph because
is a plane drawing of K_4.

Vertices, Edges, and Faces

Definition: In a plane drawing, edges divide the plane into **regions**, or **faces**.



There will always be one face with infinite area. This is called the **outside face**.

Notation. Let $p = \#$ of vertices, $q = \#$ of edges, $r = \#$ of regions. Compute the following data:

Graph	p	q	r	
Tetrahedron				
Cube				
Octahedron				
Dodecahedron				
Icosahedron				

In 1750, Euler noticed that ____________ in each of these examples.

Euler's Formula

Theorem 8.1.1 (Euler's Formula) If G is connected, then in a plane drawing of G, $p - q + r = 2$.

Proof (by induction on the number of cycles)
Base Case: If G is a tree, there is one region, so

$$p - q + r = p - (p - 1) + 1 = 2.$$

Inductive Step: Suppose that for all plane drawings with fewer than k cycles, $p - q + r = 2$, we wish to prove that in a plane drawing of a graph G with exactly k cycles, $p - q + r = 2$ also holds.

Let C be a cycle in G. Let e be any edge in C, then e is adjacent to two different regions, one inside C and one outside C.

$G \setminus e$ has fewer cycles than G, and one fewer region. The inductive hypothesis holds for $G \setminus e$, giving

Maximal Planar Graphs

A graph with "too many" edges isn't planar; how many is too many?

Goal: Find a numerical characterization of "too many"

Definition: A planar graph is called **maximal planar** if adding an edge between any two non-adjacent vertices results in a non-planar graph.

Examples. Octahedron K_4 $K_5 \setminus e$

What do we notice about these graphs?

Numerical Conditions on Planar Graphs

- Every face of a maximal planar graph is a triangle!

If not,

Theorem 8.1.2. If G is *maximal planar* and $p \geq 3$, then $q = 3p - 6$.

Proof. Consider any plane drawing of G.
Let $p = \#$ of vertices, $q = \#$ of edges, and $r = \#$ of regions.

We will count the number of face-edge incidences in two ways:

From a face-centric POV, the number of face-edge incidences is

From an edge-centric POV, the number of face-edge incidences is

Now substitute into Euler's formula:

Do we need $p \geq 3$?

Numerical Conditions on Planar Graphs

Corollary 8.1.3. Every planar graph with $p \geq 3$ vertices has at most $3p - 6$ edges.

- Start with any planar graph G with p vertices and q edges.
- Add edges to G until it is maximal planar. (with $Q \geq q$ edges.)
- This resulting graph satisfies $Q = 3p - 6$; hence $q \leq 3p - 6$.

Theorem 8.1.4. The graph K_5 is not planar.

Proof.

Numerical Conditions on Planar Graphs

Definition: The **girth** $g(G)$ of a graph G is the smallest cycle size.
Example.

*Theorem 8.1.5.** If G is planar with girth ≥ 4, then $q \leq 2p - 4$.

Proof. Modify the above proof—instead of $3r = 2q$, we know $4r \leq 2q$. This implies that

$$2 = p - q + r \leq p - q + \frac{2q}{4} = p - \frac{q}{2}.$$

Therefore, $q \leq 2p - 4$.

Theorem 8.1.5. If G is planar and bipartite, then $q \leq 2p - 4$.

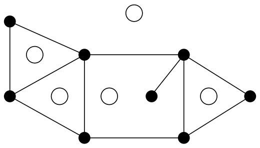
Theorem 8.1.6. $K_{3,3}$ is not planar.

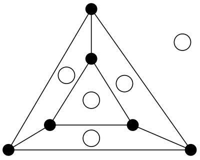
Theorem 8.1.7. Every planar graph has a vertex with degree ≤ 5.

Proof.

Dual Graphs

Definition: Given a plane drawing of a planar graph G, the **dual graph** $D(G)$ of G is a graph with vertices corresponding to the regions of G. Two vertices are connected by an edge each time the two regions share an edge as a border.



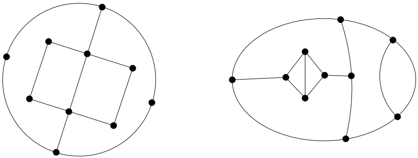


- The dual graph of a simple graph may not be simple.
  - Two regions may be adjacent multiple times.
- G and $D(G)$ have the same number of edges.

Definition: A graph G is **self-dual** if G is isomorphic to $D(G)$.

Maps

Definition: A *map* is a plane drawing of a connected, bridgeless, planar multigraph. If the map is 3-regular, then it is a **normal map**.



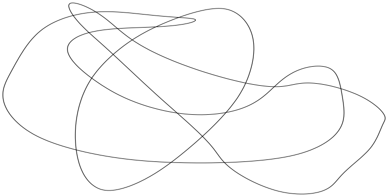
Definition: In a map, the regions are called **countries**. Countries may share several edges.

Definition: A **proper coloring** of a map is an assignment of colors to each country so that no two adjacent countries are the same color.

Question. How many colors are necessary to properly color a map?

Proper Map Colorings

Lemma 8.2.2. If M is a map that is a union of simple closed curves, the regions can be colored by two colors.



Proof. Color the regions R of M as follows:

$$\left\{\begin{matrix} \text{black} & \text{if } R \text{ is enclosed in an odd number of curves} \\ \text{white} & \text{if } R \text{ is enclosed in an even number of curves} \end{matrix}\right\}.$$

This is a proper coloring of M. Any two adjacent regions are on opposite sides of a closed curve, so the number of curves in which each is enclosed is off by one.

The Four Color Theorem

Lemma 8.2.6. (The Four Color Theorem)
Every normal map has a proper coloring by four colors.

Proof. Very hard.

$\star$ This is the wrong object $\star$

Theorem. If G is a plane drawing of a maximal planar graph, then its dual graph $D(G)$ is a normal map.

- Every face of G is a triangle $\rightsquigarrow$
- G is connected $\rightsquigarrow$
- G is planar $\rightsquigarrow$

The Four Color Theorem

Assuming Lemma 8.2.6,

$$\begin{aligned} G \text{ is maximal planar} &\Rightarrow D(G) \text{ is a normal map} \\ &\Rightarrow \text{countries of } D(G) \text{ 4-colorable} \\ &\Rightarrow \text{vertices of } G \text{ 4-colorable} \\ &\Rightarrow \chi(G) \leq 4 \end{aligned}$$

This proves

Theorem 8.2..8 If G is maximal planar, then $\chi(G) \leq 4$.

Since every planar graph is a subgraph of a maximal planar graph, Lemma C implies:

Theorem 8.2.9. If G is a planar graph, then $\chi(G) \leq 4$.

⋆ History ⋆