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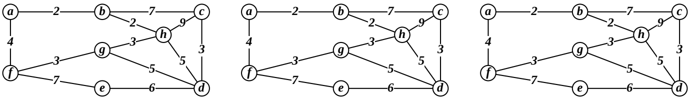

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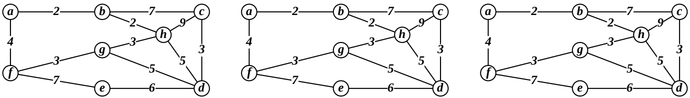

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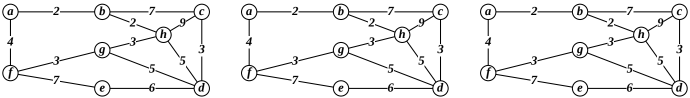

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Goal: For a weighted graph G, find a minimum-weight spanning tree.

Kruskal's algorithm

Kruskal's Algorithm finds a minimum-weight spanning tree in a weighted graph.

1. Initialization: Order the edges from lowest to highest weight:

$$w(e_1) \leq w(e_2) \leq w(e_3) \leq \cdots \leq w(e_k).$$

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   If you have a spanning tree, STOP. You have a m.w.s.t. Otherwise, continue onto step $i + 1$.

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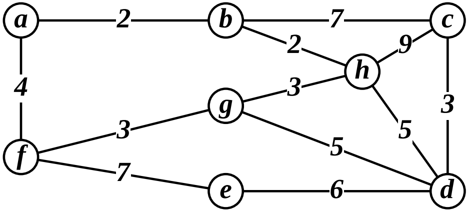

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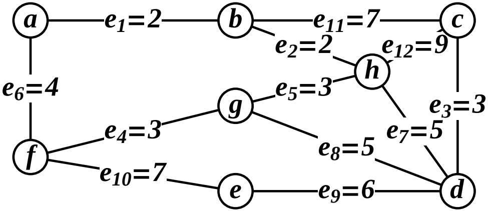

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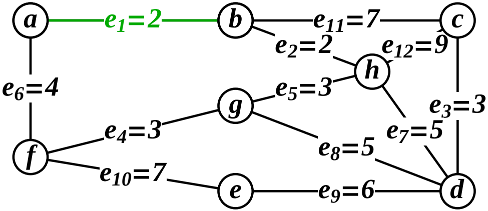

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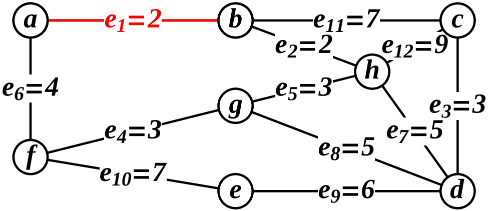

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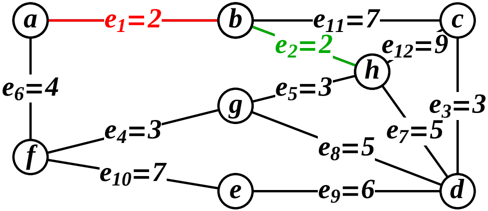

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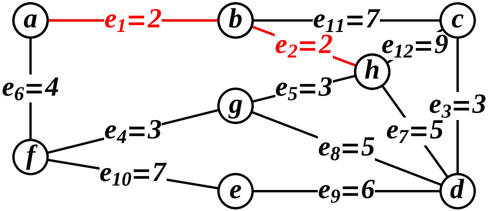

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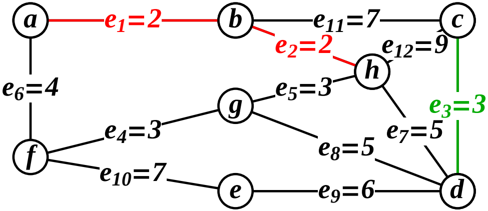

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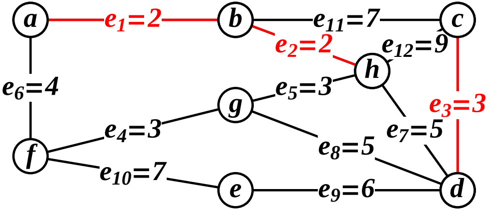

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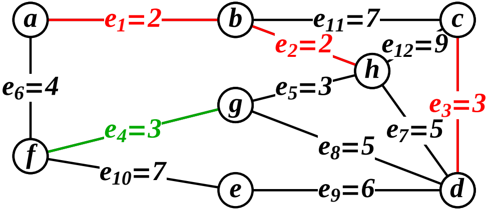

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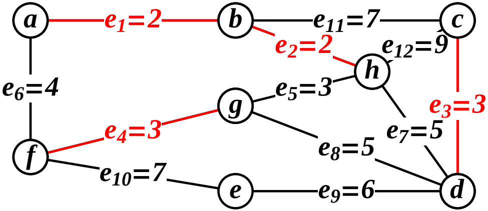

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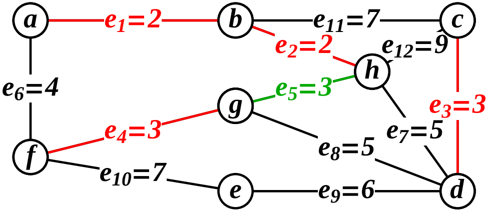

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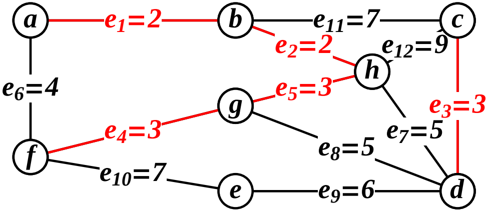

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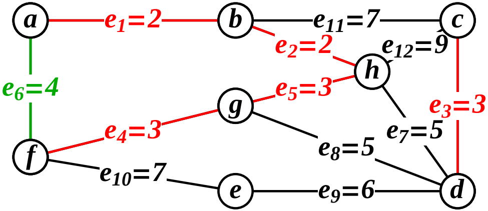

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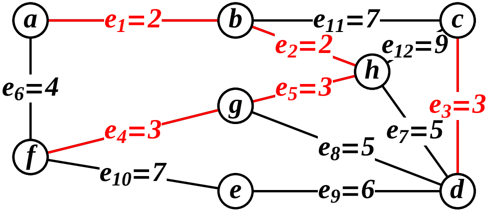

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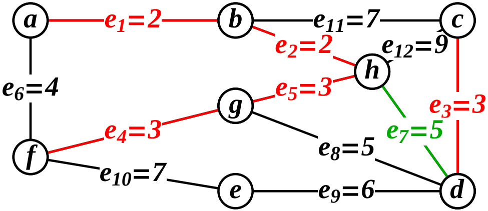

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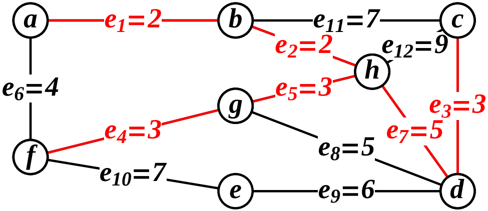

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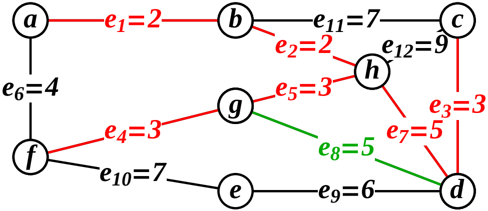

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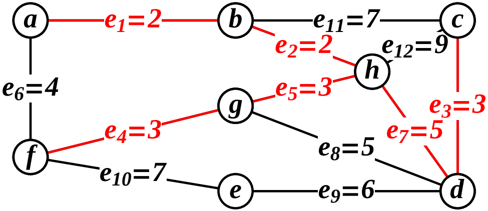

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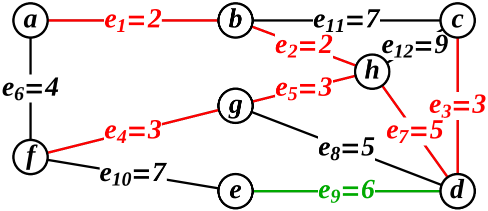

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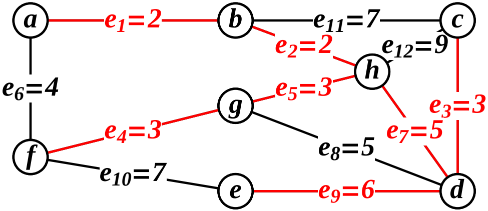

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- Greedy algorithms don't always work.

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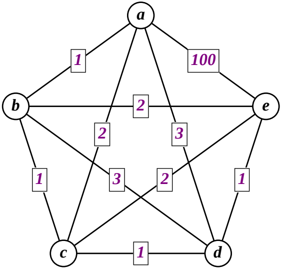

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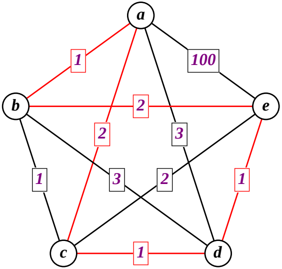

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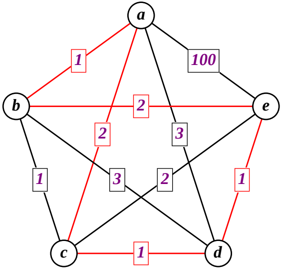

We can not use a greedy algorithm to find this **TSP tour**!

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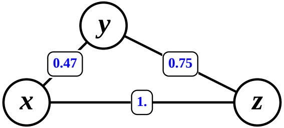
Example: Euclidean distances

Non-example: Airfares



Finding a good TSP-tour

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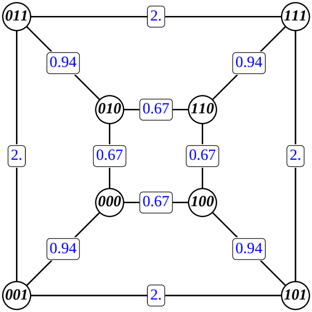

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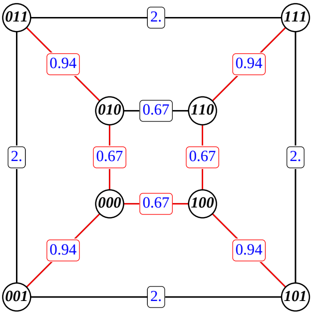

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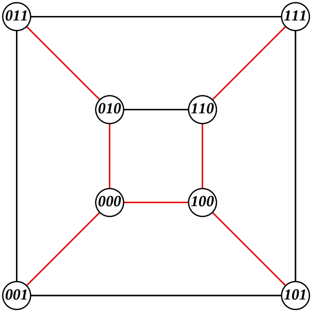

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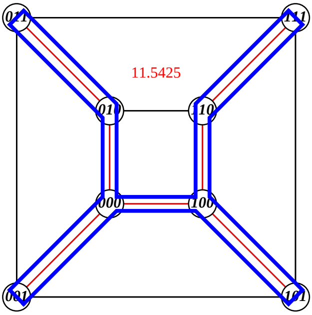

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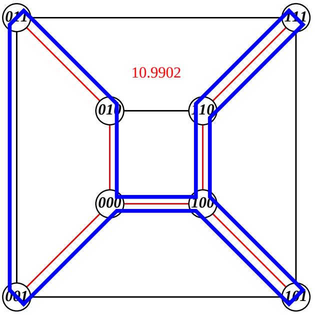

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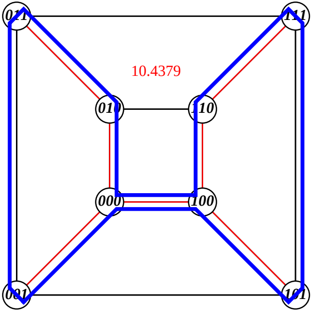

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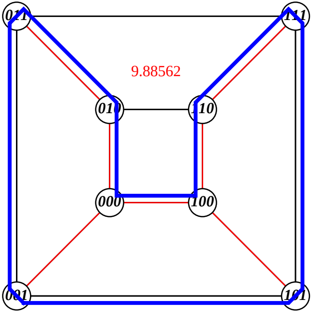

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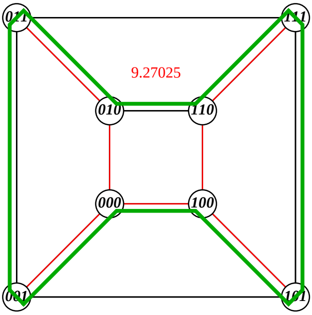

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Then,

$$\mathrm{wt}(TSP_A) \leq \mathrm{wt}(CIRC_A) = 2\,\mathrm{wt}(MST) \leq 2\,\mathrm{wt}(TSP^*).$$