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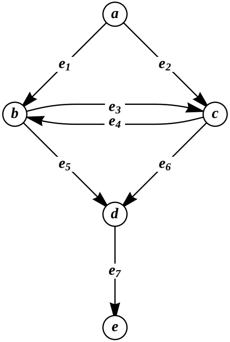

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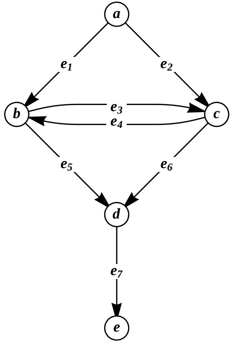

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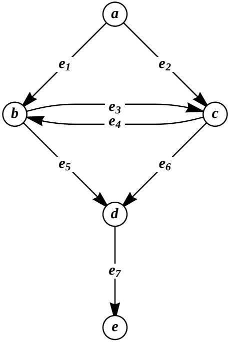

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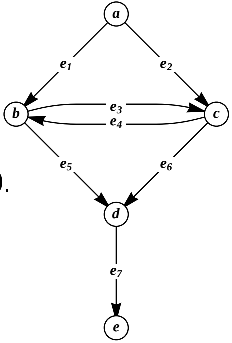

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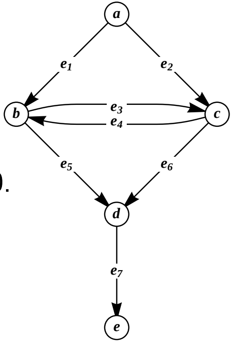
Important. Any **path** or **cycle** in a digraph must respect the direction on each edge.



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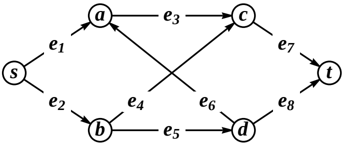

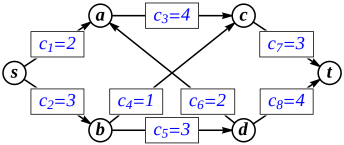

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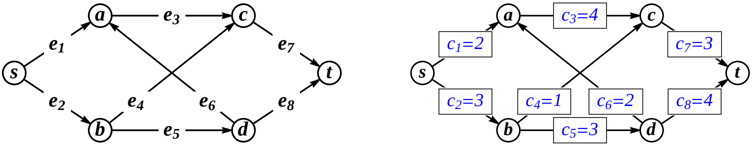

Idea. Graph networks represent real-world networks such as traffic, water, communication, etc.

Goal: Send as much "stuff" from s to t while respecting capacities.

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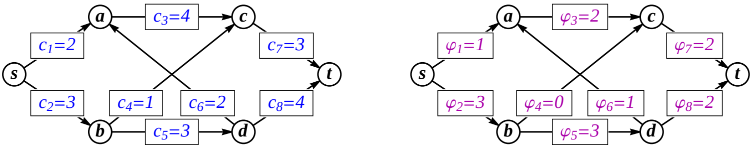

Definition: When $\varphi_e = c_e$, we say that e is **saturated**, or **at capacity**.

Maximum Flow

Theorem. Given a flow $\vec{\varphi}$ on a network G, the net flow out of s is equal to the net flow into t. Symbolically, $\sum_{e \text{ out of } s} \varphi_e = \sum_{e \text{ into } t} \varphi_e$.

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In G', flow is now conserved at every vertex except possibly t. By Kirchhoff's Global Current Law (Theorem 6.2.2), flow must be conserved at t as well.

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Definition: The **throughput** or **value** of a flow $\vec{\varphi}$ is $\sum_{e \text{ out of } s} \varphi_e$, denoted $|\vec{\varphi}|$.

Idea: The throughput is the amount of "stuff" flowing through G.

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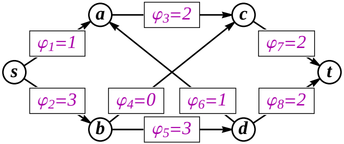

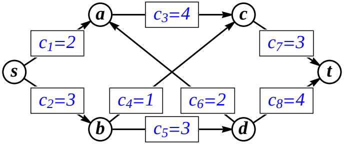

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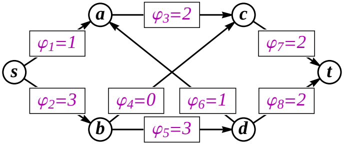

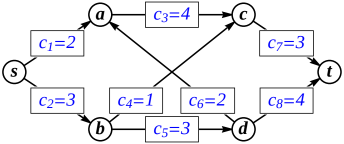

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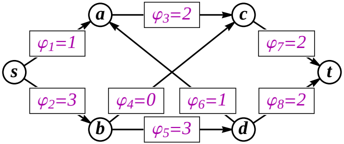

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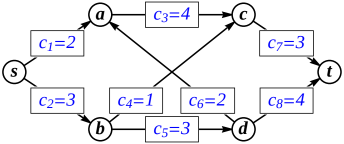

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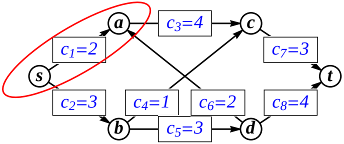

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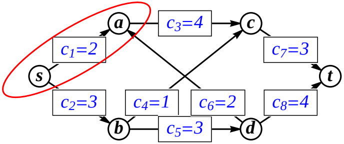

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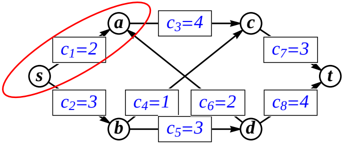

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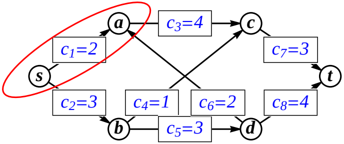

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⋆ Do **not** subtract the capacities of the edges going the other way. ⋆

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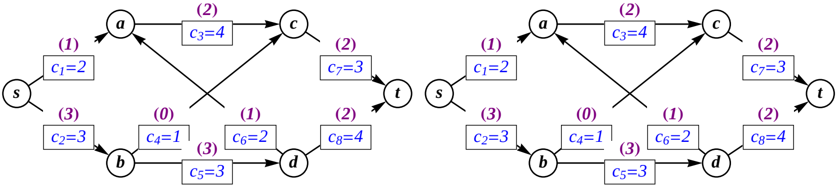

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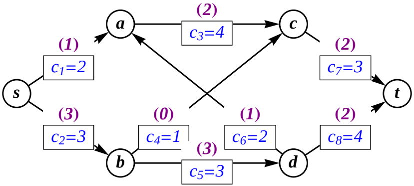

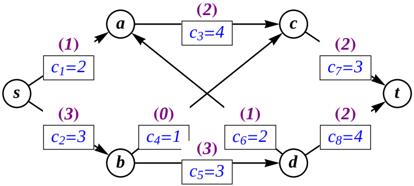

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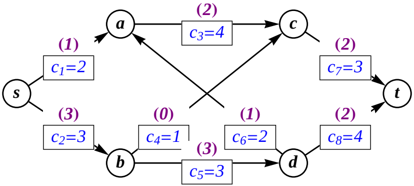

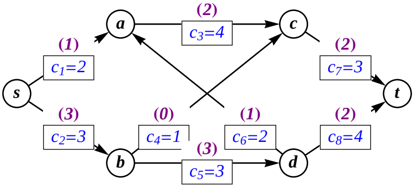

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We'll create a *companion graph* to keep track of augmenting paths.

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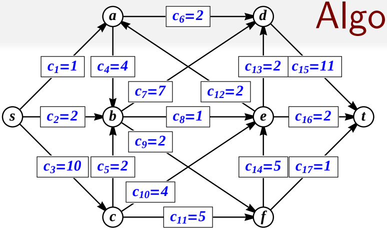
   $\longrightarrow$ Upon STOP, the current flow is a maximum flow. $\longleftarrow$
   In addition, let X be the set of vertices reachable from s in the flow companion graph. Then $[X, X^c]$ is a minimum st-cut.

A Ford–Fulkerson Algorithm Example



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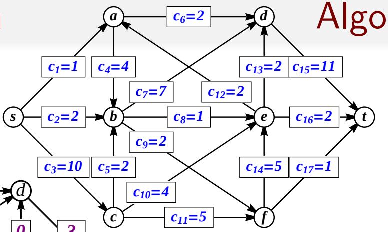

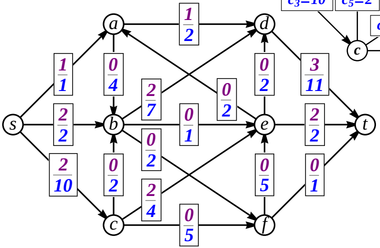

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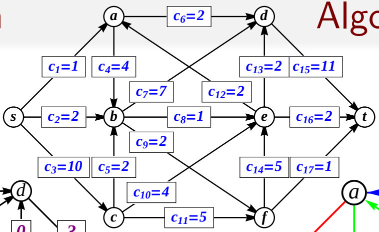

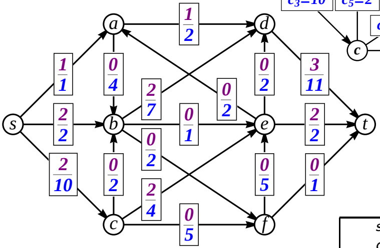

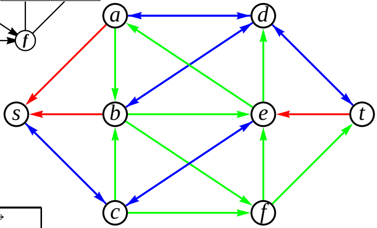

$s \longrightarrow c \longrightarrow b \longrightarrow$ $d \longrightarrow t$: 2 units
$s \longrightarrow c \longrightarrow f \longrightarrow$ $e \longrightarrow d \longrightarrow t$: 2 units

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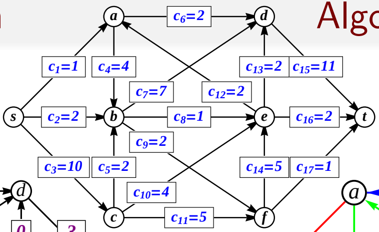

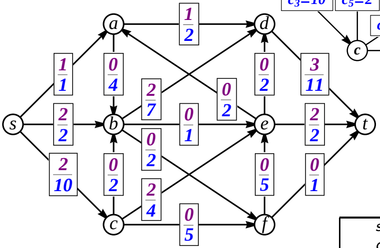

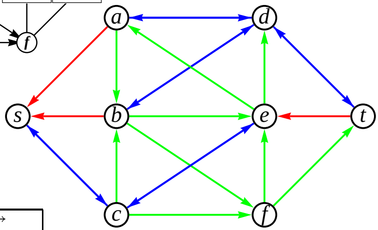

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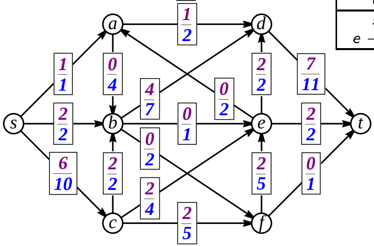

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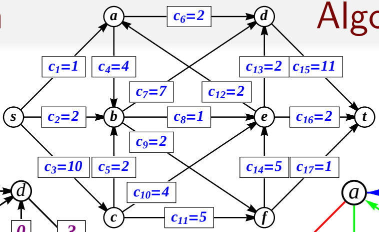

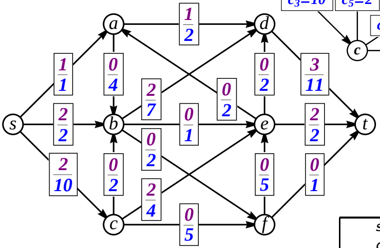

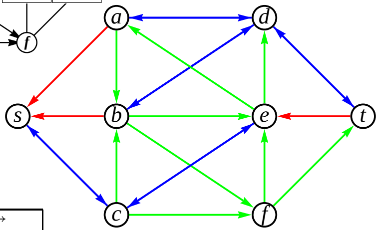

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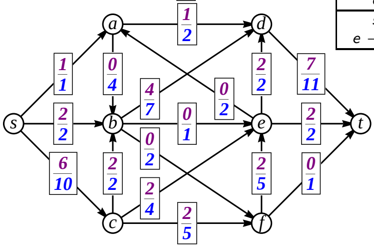

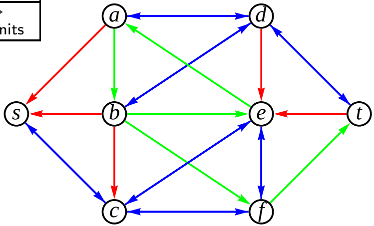

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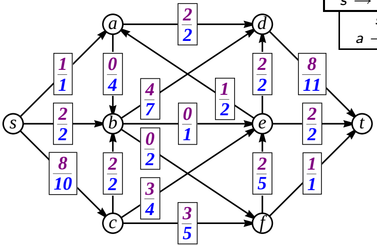

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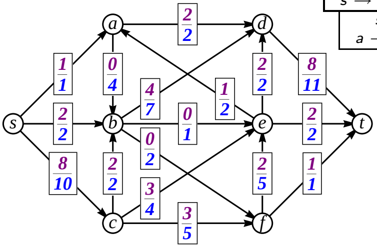

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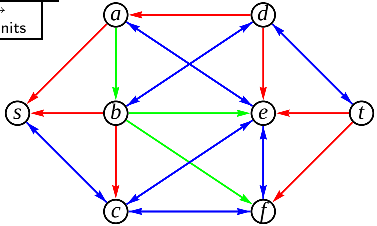

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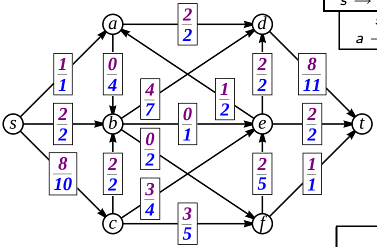

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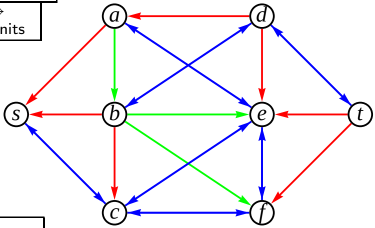

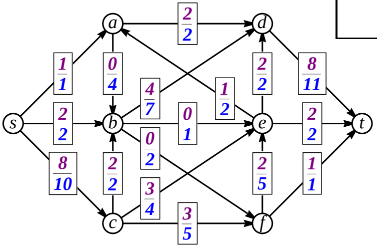

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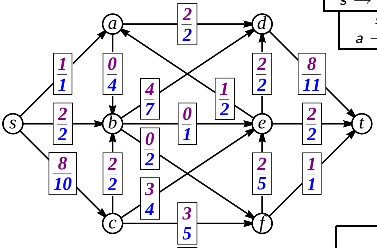

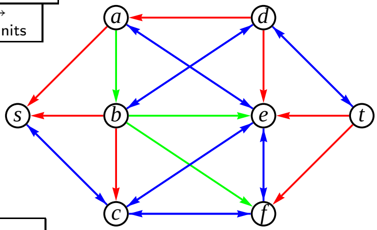

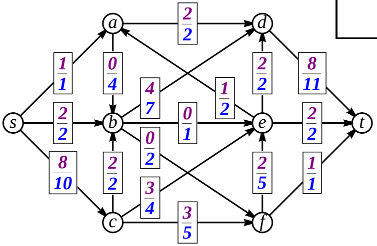

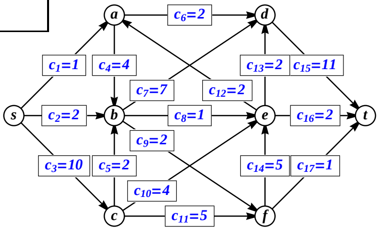

$X = \{\underline{\qquad\qquad}\}$, $[X, X^c] = \{\underline{\qquad\qquad\qquad}\}$, and $\left|[X, X^c]\right| = \underline{\qquad}$.

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Conclusion. The flow is a max flow and the st-cut is a min cut.

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- When the capacities are integers, we always increase the throughput by integers. The algorithm does work when the capacities are not integers, but the proof is more involved.
- As presented here, this algorithm may be very slow.