Stable Marriages

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Men's Preferences

	Bob	Doug	Fred
1st	Alice	Alice	Elena
2nd	Clara	Elena	Clara
3rd	Elena	Clara	Alice

Women's Preferences

	Alice	Clara	Elena
1st	Fred	Bob	Doug
2nd	Bob	Fred	Fred
3rd	Doug	Doug	Bob

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- Bob prefers Clara to Alice

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- Alice is married to Bob
- Clara is married to Doug
- Bob prefers Clara to Alice

If Clara prefers Bob to Doug: ____________________________

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Example. Suppose:

- Alice is married to Bob
- Clara is married to Doug
- Bob prefers Clara to Alice

If Clara prefers Bob to Doug: ____________________

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$<<$Time for your moment of zen$>>$

Applying the Gale–Shapley Algorithm

Here is a complete set of preferences for 4 men and 4 women.

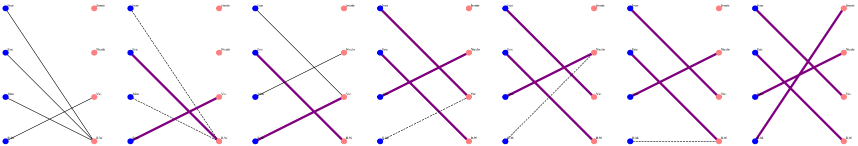
Men's Preferences

	Ivan	Eric	John	R.M.
1^{st}	R.W.	R.W.	R.W.	Vic.
2^{nd}	Vic.	Jennie	Nicole	Nicole
3^{rd}	Jennie	Vic.	Jennie	R.W.
4^{th}	Nicole	Nicole	Vic.	Jennie

Women's Preferences

	Jennie	Nicole	Vic.	R.W.
1^{st}	Eric	John	John	Eric
2^{nd}	John	R.M.	Ivan	R.M.
3^{rd}	R.M.	Eric	R.M.	Ivan
4^{th}	Ivan	Ivan	Eric	John

The Algorithm, Pictorially



Men's Preferences

Ivan	Eric	John	R.M.
R.W.	R.W.	R.W.	Vic.
Vic.	Jennie	Nicole	Nicole
Jennie	Vic.	Jennie	R.W.
Nicole	Nicole	Vic.	Jennie

Women's Preferences

Jennie	Nicole	Vic.	R.W.
Eric	John	John	Eric
John	R.M.	Ivan	R.M.
R.M.	Eric	R.M.	Ivan
Ivan	Ivan	Eric	John

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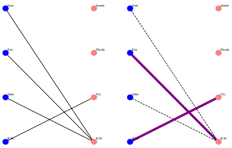

Men's Preferences

Ivan	Eric	John	R.M.
R.W.	**R.W.**	**R.W.**	**Vic.**
Vic.	Jennie	Nicole	Nicole
Jennie	Vic.	Jennie	R.W.
Nicole	Nicole	Vic.	Jennie

Women's Preferences

Jennie	Nicole	Vic.	R.W.
Eric	John	John	**Eric**
John	R.M.	Ivan	R.M.
R.M.	Eric	**R.M.**	**Ivan**
Ivan	Ivan	Eric	**John**

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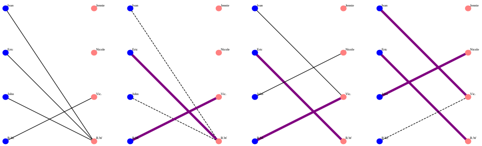

Men's Preferences

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R.W.	**R.W.**	R.W.	**Vic.**
Vic.	Jennie	**Nicole**	Nicole
Jennie	Vic.	Jennie	R.W.
Nicole	Nicole	Vic.	Jennie

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John	R.M.	**Ivan**	R.M.
R.M.	Eric	**R.M.**	Ivan
Ivan	Ivan	Eric	John

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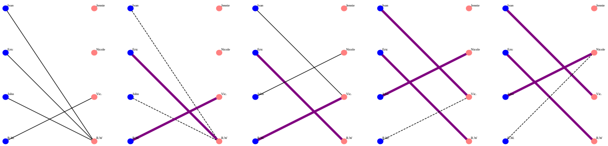

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Nicole	Nicole	Vic.	Jennie

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R.M.	Eric	R.M.	Ivan
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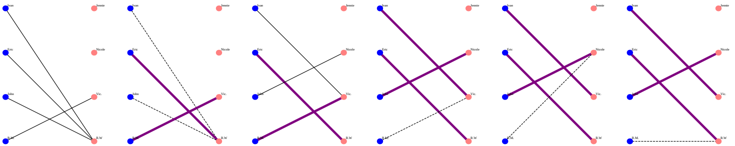

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R.M.	Eric	R.M.	Ivan
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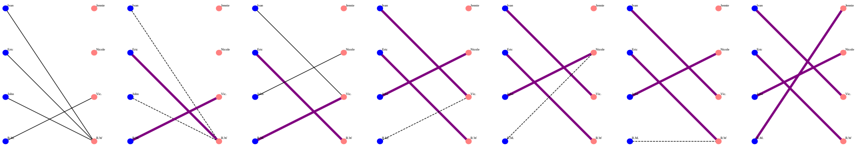

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R.M.	Eric	R.M.	Ivan
Ivan	Ivan	Eric	John

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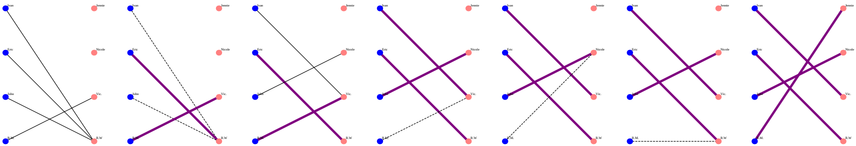

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- there are a finite number to be made.

Claim: Upon termination, everyone is engaged.

- Once a woman has been proposed to, she stays engaged.
- If a woman is not engaged at the end, she had no proposal.
- It follows that there is also some man not engaged; however, he must have proposed to the unengaged woman during some round!

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- Therefore, there is no instability.

Male-optimality

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- Let M be the first man who is rejected by his optimal woman W during the algorithm.

M-Optimal Pairings ($\mathcal{S}'$)

Relative Pref's

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- M is rejected because some man N proposes to W whom W prefers to M.
- Since M is the *first* man rejected, we know N likes W at least as much as his optimal woman.
- This, in turn, creates an instability in $\mathcal{S}'$ since W prefers N to M and N prefers W to the woman he is paired with.

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- If not all rankings are made, then there may be unmatched people. For example, what if Robot Man did not like Jennie?
- The National Resident Matching Program (`http://www.nrmp.org`) implements this algorithm to match medical students to residency programs.