Stable Marriages

Let's play matchmaker.

Given n men and n women, where

- Each man has a complete list of preferences for the women,
- Each woman has a complete list of preferences for the men,

Men's Preferences

	Bob	Doug	Fred
1st	Alice	Alice	Elena
2nd	Clara	Elena	Clara
3rd	Elena	Clara	Alice

Women's Preferences

	Alice	Clara	Elena
1st	Fred	Bob	Doug
2nd	Bob	Fred	Fred
3rd	Doug	Doug	Bob

Stable Marriages

Goal: Create a perfect matching of **stable marriages**.

That is, find a set of n couples where there are no instabilities:

Definition: An **instability** is when one man and one woman both prefer each other to their spouses.

Example. Suppose:

- Alice is married to Bob
- Clara is married to Doug
- Bob prefers Clara to Alice

If Clara prefers Bob to Doug: _____

If Clara prefers Doug to Bob: _____

The Gale–Shapley Algorithm

Theorem. (Gale, Shapley, 1962) In the above situation, there always exists a perfect matching of stable marriages.

Proof. Use the **Gale–Shapley Algorithm** to create the marriages.

1. Start with no couples engaged.
2. As long as at least one man is unengaged, repeat the following: Each unengaged man proposes to his next most preferred woman. Each woman then decides whether to accept or reject the proposal(s), as follows:
   - If she is not already engaged, she accepts the proposal.
   - If she is already engaged, she uses her preference list; she accepts the proposal of the man she prefers the most and rejects all others.
3. When all men are engaged, stop. There are n stable marriages.

<<Time for your moment of zen>>

Applying the Gale–Shapley Algorithm

Here is a complete set of preferences for 4 men and 4 women.

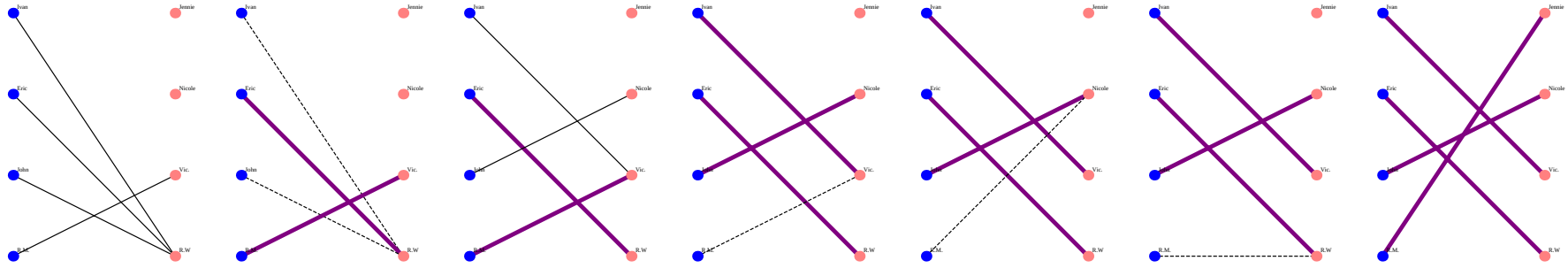
Men's Preferences

	Ivan	Eric	John	R.M.
1st	R.W.	R.W.	R.W.	Vic.
2nd	Vic.	Jennie	Nicole	Nicole
3rd	Jennie	Vic.	Jennie	R.W.
4th	Nicole	Nicole	Vic.	Jennie

Women's Preferences

	Jennie	Nicole	Vic.	R.W.
1st	Eric	John	John	Eric
2nd	John	R.M.	Ivan	R.M.
3rd	R.M.	Eric	R.M.	Ivan
4th	Ivan	Ivan	Eric	John

The Algorithm, Pictorially



Men's Preferences

Ivan	Eric	John	R.M.
R.W.	R.W.	R.W.	Vic.
Vic.	Jennie	Nicole	Nicole
Jennie	Vic.	Jennie	R.W.
Nicole	Nicole	Vic.	Jennie

Women's Preferences

Jennie	Nicole	Vic.	R.W.
Eric	John	John	Eric
John	R.M.	Ivan	R.M.
R.M.	Eric	R.M.	Ivan
Ivan	Ivan	Eric	John

Proof of Correctness

Claim. The Gale–Shapley Algorithm gives a set of n stable marriages.

Proof. We must show that the algorithm always stops, and that when it stops, the output is indeed a full set of stable marriages.

The algorithm terminates.

- In each step, at least one proposal occurs
- there are a finite number to be made.

Claim: Upon termination, everyone is engaged:

- Once a woman has been proposed to, she stays engaged.
- If a woman is not engaged at the end, she had no proposal.
- It follows that there is also some man not engaged; however, he must have proposed to the unengaged woman during some round!

Proof of Correctness

The output is a set of stable marriages.

We ask: Is there an instability?

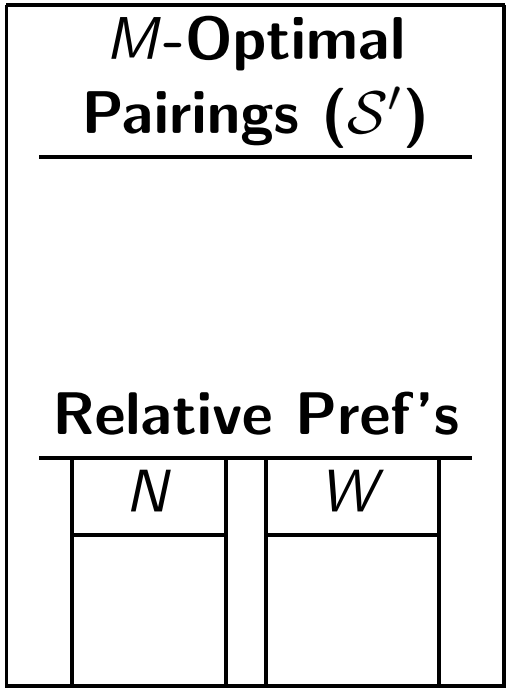
- Suppose Bob prefers Clara to his current wife.
- Bob must have proposed to Clara before his current wife.
- Clara must have turned down Bob.
  - Clara was proposed to by someone she prefers!
- Hence, Clara must prefer her current husband to Bob.
- Therefore, there is no instability.

Male-optimality

Claim. The marriages $\mathcal{S}$ generated by the Gale–Shapley Algorithm are **male optimal**. That is, given any other set of stable marriages, each man will only be paired with a woman lower on his preference list.

Proof. Suppose that during the Gale–Shapley Algorithm, there is a man who is paired with a "sub-optimal" woman.

- Let M be the first man who is rejected by his optimal woman W during the algorithm. [*That is, there is some other set $\mathcal{S}'$ of stable marriages in which M is paired with W.*]
- M is rejected because some man N proposes to W whom W prefers to M.
- Since M is the *first* man rejected, we know N likes W at least as much as his optimal woman.
- This, in turn, creates an instability in $\mathcal{S}'$ since W prefers N to M and N prefers W to the woman he is paired with.



Last remarks

- The marriages generated by Gale–Shapley are male optimal.
- The marriages generated by Gale–Shapley are female pessimal.
- Run the algorithm with the women proposing to reverse the roles.

  If you do this and get the same marriages, _____
- If not all rankings are made, then there may be unmatched people. For example, what if Robot Man did not like Jennie?
- The National Resident Matching Program (`http://www.nrmp.org`) implements this algorithm to match medical students to residency programs.