Algorithms

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To verify the **correctness** of an algorithm:

1. Verify that the algorithm terminates. (often invoking finiteness)

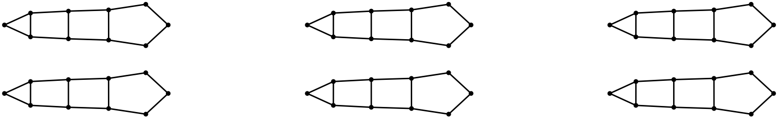
2. Verify that the result satisfies the desired conditions.

Matchings in Graphs

Definition: A **matching** M in a graph G is a subset of edges of G that share no vertices.

Definition: A **maximal matching** M is a matching such that the inclusion into M of any edge of $G \setminus M$ is no longer a matching.

Definition: A **maximum matching** is a matching M that has the most edges possible for the graph G.



Thought Exercise: What is the result of overlapping two matchings?

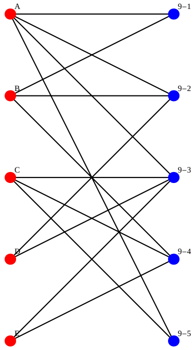
Recall. A **perfect matching** is a matching involving every vertex of G.

⋆ We will discuss matchings in a bipartite graph ⋆

Application: Scheduling

Suppose you are working in a group trying to complete all the problems on the homework. Depending on everyone's preferences, you would like to assign each member one problem to do.

Person A likes problems 9-1, 9-2, 9-3, and 9-5.
Person B likes problems 9-1, 9-2, and 9-4.
Person C likes problems 9-3, 9-4, and 9-5.
Person D likes problems 9-2 and 9-3.
Person E likes problems 9-3 and 9-4.



Create a graph that models the situation.

Question:
What is a maximum matching for this graph?

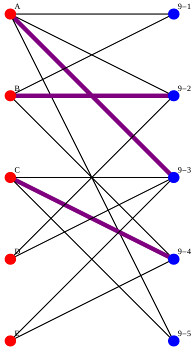
We will use an algorithm to answer this question.

Motivating The Hungarian Algorithm

Let us work through the basic idea behind the algorithm.

We start with an initial matching; we might as well make it maximal.

Why is the pictured matching maximal?



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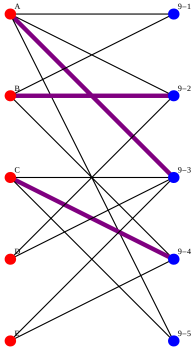

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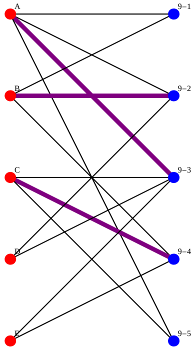

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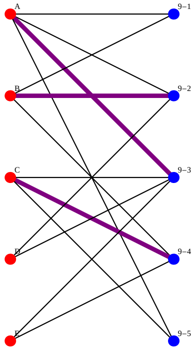

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Definition: An M-**augmenting path** is an M-alternating path that begins AND ends at unmatched vertices.

It is *augmenting* because we can improve M by toggling the edges between those in M and those not in M.

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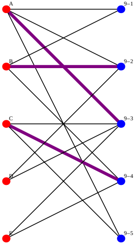

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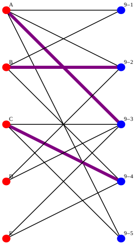

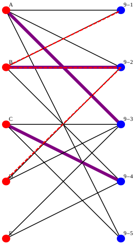

Given M, $P = D \to$ 9-2 $\to B \to$ 9-1 is an M-augmenting path. Toggling the edges in P gives a new matching M'.

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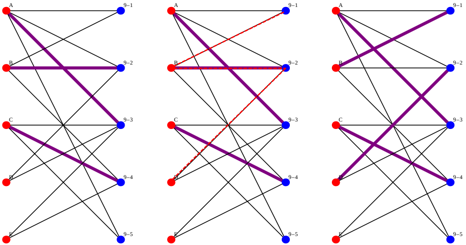

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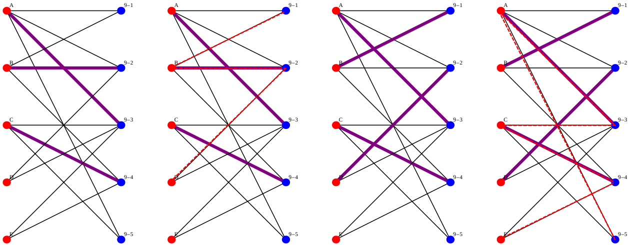

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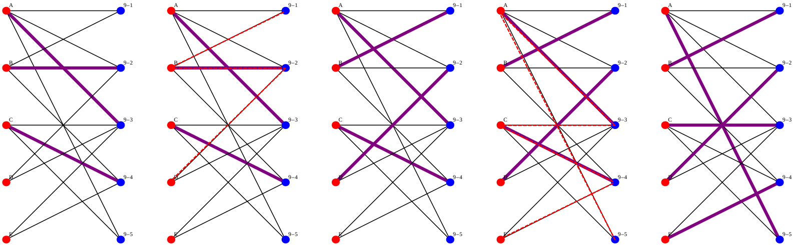

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The matching M'' is maximal.

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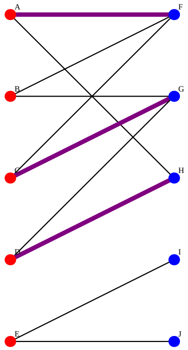
Return to Step 2.

Applying the Hungarian Algorithm

Here is something that might happen during an application of the Hungarian algorithm:

Example. There is no M-augmenting path starting at B in the graph to the right.



We would mark B ineligible and move on to the next eligible, unmatched red vertex in the graph (E).

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Claim. The Hungarian Algorithm gives a maximum matching.

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This path is an M-augmenting path, contradicting the definition of M.