(Vertex) Colorings

Definition: A **coloring** of a graph G is a labeling of the vertices of G with colors. [Technically, it is a function $f : V(G) \to \{1, 2, \ldots, l\}$.]

Definition: A **proper coloring** of G is a coloring of G such that no two adjacent vertices are labeled with the same color.

Example: W_6:

We can properly color W_6 with ____ colors and no fewer.

Of interest: What is the fewest colors necessary to properly color G?

The chromatic number of a graph

Definition: The minimum number of colors necessary to properly color a graph G is called the **chromatic number** of G, denoted $\chi(G) =$ "chi".

Example: $\chi(K_n) =$ ____

Proof: In order to have a proper coloring of K_n, we would need to use at least ____ colors, because every vertex is adjacent to every other vertex. With fewer than ____ colors, there would be two adjacent vertices colored the same. And indeed, placing a different color on each vertex is a proper coloring of K_n.

⋆ $\chi(G) = k$ is the same as:

1. There is a proper coloring of G with k colors. (Show it!)
2. There is no proper coloring of G with $k - 1$ colors. (Prove it!)

Chromatic numbers and subgraphs

Lemma C: If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Proof: If $\chi(G) = k$, then there is a proper coloring of G using k colors. Let the vertices of H inherit their coloring from G. This gives a proper coloring of H using k colors, which implies $\chi(H) \leq k$.

Corollary: For any graph G, $\chi(G) \geq \omega(G)$.

Proof: Apply Lemma C to the subgraph of G isomorphic to $K_{\omega(G)}$.

Example: Calculate $\chi(G)$ for this graph G:

Critical graphs

One way to prove that G can not be properly colored with $k-1$ colors is to find a subgraph H of G that requires k colors.

How small can this subgraph be?

Definition: A graph G is called **critical** if for every proper subgraph $H \subsetneq G$, then $\chi(H) < \chi(G)$.

Theorem 2.1.2: Every graph G contains a critical subgraph H such that $\chi(H) = \chi(G)$.

Proof: If G is critical, stop. Define $H = G$.
If not, then there exists a proper subgraph G_1 of G with ________.
If G_1 is critical, stop. Define $H = G_1$.
If not, then there exists a proper subgraph G_2 of G_1 with ______ $\cdots$
Since G is finite, there will be some proper subgraph G_l of G_{l-1} such that G_l is critical and $\chi(G_l) = \chi(G_{l-1}) = \cdots = \chi(G)$.

Critical graphs

What do we know about critical graphs?

Theorem 2.1.1: Every critical graph is connected.

Theorem 2.1.3: If G is critical with $\chi(G) = 4$, then for all $v \in V(G)$, $\deg(v) \geq 3$.

Proof by contradiction: Suppose not. Then there is some $v \in V(G)$ with $\deg(v) \leq 2$. Remove v from G to create H.

Similarly: If G is critical, then for all $v \in V(G)$, $\deg(v) \geq \chi(G) - 1$.

Bipartite graphs

Question: What is $\chi(C_n)$ when n is odd?

Answer:

Definition: A graph is called **bipartite** if $\chi(G) \leq 2$.

Examples: $K_{m,n}$, $\square_n$, Trees

Theorem 2.1.6: G is bipartite $\iff$ every cycle in G has even length.

$(\Rightarrow)$ Let G be bipartite. Assume that there is some cycle C of odd length contained in G . . .

Proof of Theorem 2.1.6

($\Leftarrow$) Suppose that every cycle in G has even length. We want to show that G is bipartite. Consider the case when G is connected.

Plan: Construct a coloring on G and prove that it is proper.

Choose some starting vertex x and color it blue. For every other vertex y, calculate the distance from y to x and then color y:

$$\begin{cases} \text{blue} & \text{if } d(x,y) \text{ is even.} \\ \text{red} & \text{if } d(x,y) \text{ is odd.} \end{cases}$$

Question: Is this a proper coloring of G?

Suppose not. Then there are two vertices v and w of the same color that are adjacent. This generates a contradiction because there exists an odd cycle as follows:

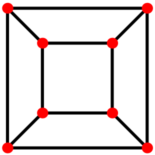
Edge Coloring

Parallel to the idea of vertex coloring is the idea of edge coloring.

Definition: An **edge coloring** of a graph G is a labeling of the edges of G with colors. [Technically, it is a function $f : E(G) \to \{1, 2, \ldots, l\}$.]

Definition: A **proper** edge coloring of G is an edge coloring of G such that no two *adjacent edges* are colored the same.

Example: Cube graph ($\square_3$):



We can properly edge color $\square_3$ with ____ colors and no fewer.

Definition: The minimum number of colors necessary to properly edge color a graph G is called the **edge chromatic number** of G, denoted $\chi'(G) =$ "chi prime".

Edge coloring theorems

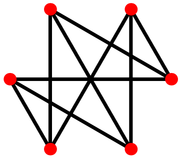
Theorem 2.2.1: For any graph G, $\chi'(G) \geq \Delta(G)$.

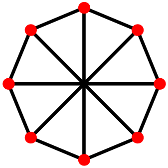
Theorem 2.2.2: Vizing's Theorem:
For any graph G, $\chi'(G)$ equals either $\Delta(G)$ or $\Delta(G) + 1$.

Proof: Hard. (See reference [24] if interested.)

Consequence: To determine $\chi'(G)$,

Fact: **Most** 3-regular graphs have edge chromatic number 3.



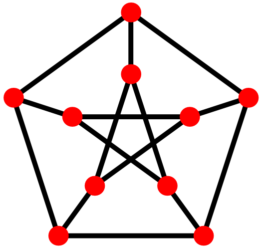


Snarks

Definition: Another name for 3-regular is **cubic**.

Definition: A 3-regular graph with edge chromatic number 4 is called a **snark**.

Example: The Petersen graph P:



The edge chromatic number of complete graphs

Goal: Determine $\chi'(K_n)$ for all n.

Vertex Degree Analysis: The degree of every vertex in K_n is ____.

Vizing's theorem implies that $\chi'(K_n) =$____ or ____.

If $\chi'(K_n) =$____, then each vertex has an edge leaving of each color.

Q: How many red edges are there?

This is only an integer when:

So, the best we can expect is that $\begin{cases} \chi'(K_{2n}) = \\ \chi'(K_{2n-1}) = \end{cases}$

The edge chromatic number of complete graphs

Theorem 2.2.3: $\chi'(K_{2n}) = 2n - 1$.

Proof: We prove this using the *turning trick*.

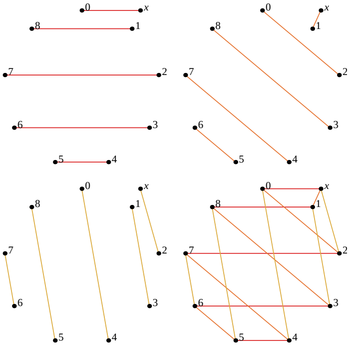
Label the vertices of K_{2n} $0, 1, \ldots, 2n-2, x$. Now,

Connect 0 with x,

Connect 1 with $2n-2$,

$\vdots$

Connect $n-1$ with n.



Now **turn** the edges.

And do it again. (and again, ...)

Each time, new edges are used. This is because each of the edges is a different "circular length": vertices are at circ. distance 1, 3, 5, ..., 4, 2 from each other, and x is connected to a different vertex each time.

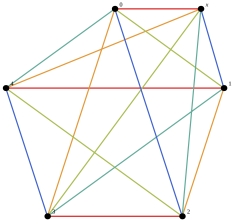
The edge chromatic number of complete graphs

Theorem 2.2.4: $\chi'(K_{2n-1}) = 2n - 1$.

This construction also gives a way to edge color K_{2n-1} with $2n - 1$ colors—simply delete vertex x!

This is related to the area of combinatorial designs.
Question: Is it possible for six tennis players to play one match per day in a five-day tournament in such a way that each player plays each other player once?

Day 1	0x	14	23
Day 2	1x	20	34
Day 3	2x	31	40
Day 4	3x	42	01
Day 5	4x	03	12



Theorem 2.2.3 proves there is such a tournament for all even numbers.