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L'Hôpital's rule. If $f(x)$, $g(x)$ differentiable, $g'(x) \neq 0$ on $I \setminus \{x_0\}$, and $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, and $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists, then

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