Indefinite integrals

Try non-standard substitutions, don't forget $+C$

Example. Calculate $I_1 = \int \frac{\sin x}{\sin x + \cos x}\,dx$ (p.148)

Solution. Consider $I_2 = \int \frac{\cos x}{\sin x + \cos x}\,dx$.

$$I_1 + I_2 = \int \frac{\sin x + \cos x}{\sin x + \cos x}\,dx = \int 1\,dx = x + C_1.$$

$$I_2 - I_1 = \int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx = \int \frac{du}{u} = \ln(\sin x + \cos x) + C_2.$$

Solving the system of equations, $I_1 = \frac{1}{2}x - \frac{1}{2}\ln(\sin x + \cos x) + C$.

Example. For $a > 0$, solve $\int \frac{1}{x\sqrt{x^{2a} + x^a + 1}}\,dx$ (p.148)

Solution. Rewrite, complete the square, substitute $u = \frac{1}{x^a} + \frac{1}{2}$:

$$\int \frac{1}{xx^a\sqrt{1 + \frac{1}{x^a} + \frac{1}{x^{2a}}}}\,dx = \int \frac{1}{\sqrt{(\frac{1}{x^a} + \frac{1}{2})^2 + \frac{3}{4}}} \cdot \frac{dx}{x^{a+1}} = \int \frac{1}{\sqrt{u^2 + 3/4}} \cdot \frac{du}{-a}.$$

Solve w/trig. subs.: $\frac{-1}{a}\ln(u + \sqrt{u^2 + 3/4}) + C$, and resub for u.

Definite integrals

The bounds of the integral are key. Why are they what they are?

Example. Prove $\int_0^{\pi} xf(\sin x)\,dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x)\,dx.$ (p.150)

Split the LHS: $\int_0^{\pi} xf(\sin x)\,dx = \int_0^{\frac{\pi}{2}} xf(\sin x)\,dx + \int_{\frac{\pi}{2}}^{\pi} xf(\sin x)\,dx$

$= \int_0^{\frac{\pi}{2}} xf(\sin x)\,dx + \int_0^{\frac{\pi}{2}} (\pi - x)f(\sin(\pi - x))\,dx$

$= \int_0^{\frac{\pi}{2}} xf(\sin x)\,dx - \int_0^{\frac{\pi}{2}} xf(\sin x)\,dx + \pi \int_0^{\frac{\pi}{2}} f(\sin x)\,dx$

Use a definite integral to calculate a limit by way of a Riemann sum.

Example. Find $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)$. (p.153)

Solution. Rewrite as $\lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}\right)\left(\frac{1}{n}\right)$.

Recognize this as the Riemann sum for $\int_{x=0}^{x=1} \frac{1}{1+x}\,dx$.

Evaluating the integral gives $\ln(1+x)\big|_0^1 = \ln(2) - \ln(1) = \ln(2)$.

Important Integral Inequalities (III)

Positivity. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.

If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

Cauchy-Schwarz. $\left(\int_D f(x)g(x)\,dx\right)^2 \leq \left(\int_D f(x)\,dx\right)\left(\int_D g(x)\,dx\right)$

Minkowski. For $p > 1$,
$\left(\int_D |f(x)+g(x)|^p\,dx\right)^{\frac{1}{p}} \leq \left(\int_D |f(x)|^p\,dx\right)^{\frac{1}{p}} + \left(\int_D |g(x)|^p\,dx\right)^{\frac{1}{p}}$

Hölder. If $p, q > 1$ satisfy $1/p + 1/q = 1$, then
$\left(\int_D |f(x)g(x)|\,dx\right) \leq \left(\int_D |f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_D |g(x)|^q\,dx\right)^{\frac{1}{q}}$

Chebyshev. Let f, g be increasing functions. Then for all $a < b$:
$(b-a)\int_a^b f(x)g(x)\,dx \geq \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$