Course Notes

Putnam Preparation, Fall 2011

Queens College, Math 390

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On the web: `http://people.qc.cuny.edu/faculty/christopher.hanusa/courses/390fa11/`

Reference List

The following are references that I recommend to complement this course. These books are available in the library:

The William Lowell Putnam Mathematical Competition 1938–1964
The William Lowell Putnam Mathematical Competition 1985–2000
Larson. *Problem solving through problems*

Also:
Gelca and Andreescu. *Putnam and Beyond.* (online access)
The William Lowell Putnam Mathematical Competition 1965–1984

See the course website for additional links.

Course Expectations

- **Daily schedule:**
  - Go over homework in groups. (15–30 min)
  - Introduction to the day's topic. (15–30 min)
  - Problem solving. (60–90 min)
  - Problem presentations (30–45 min)
  - *Homework:* Write up two problems **really well**.
- **Put in the time.**
  - Work together outside class.
  - Bounce proof ideas around.
  - Help each other with the writeup.
  - Three credits = (at least) nine hours / week out of class.
- **Stay in contact.**
  - If you are confused, ask questions (in class and out).

Week 1

This week we will focus on Sections 1.1–1.3 of *Putnam and Beyond*, which discusses proof techniques.

Proof by contradiction

Induction

The Pigeonhole Principle: If $kn+1$ objects are distributed among n boxes, one of the boxes will contain at least $k+1$ objects.

Proof by contradiction

Euclid's theorem. There are infinitely many prime numbers.

Proof. Suppose not.

Example. Prove that there is no polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

with integer coefficients and of degree at least 1 with the property that $P(0)$, $P(1)$, $P(2)$, ... are all prime numbers.

Proof. Suppose not. Then such a polynomial $P(x)$ exists.

Induction proofs

The principle of induction Given $P(n)$, a property depending on a positive integer n,

1. if $P(n_0)$ is true for some positive integer n_0, and
2. if for every $k \geq n_0$, $P(k)$ true implies $P(k+1)$ true,

then $P(n)$ is true for all $n \geq n_0$.

[**Strong induction:** use $P(n_0), \ldots, P(k)$ to prove $P(k+1)$ true.]

Fermat's little theorem. Let p be a prime number, and n a positive integer. Then $n^p - n$ is divisible by p.

The Pigeonhole Principle

The Pigeonhole Principle: If $kn + 1$ objects are distributed among n boxes, one of the boxes will contain at least $k + 1$ objects.

- Used to show **existence**.
- Is more powerful than first appears.
- **Key concept:** Determine the boxes, determine the objects.

Examples of solvable questions:

- Show that if $n + 1$ integers are chosen from the set $\{1, 2, \dots, 3n\}$, then there are always two integers which differ by at most two.
- Show that every sequence $a_1, a_2, \dots, a_{n^2+1}$ of distinct real numbers contains a monotonic subsequence of length $n + 1$.
  - Suppose that there is no increasing subsequence of length $n + 1$. Determine the length l_i of the longest increasing subsequence starting with a_i.
  - The values l_i (the objects) are between 1 and n (the boxes).