Vectors

We will be using vectors and matrices to store and manipulate data.

Definition: A **vector** $\vec{\mathbf{v}}$ is a column of numbers. Use bold faced letters or vector signs to distinguish vectors from other variables.

We refer to the **entries** of a vector by using subscripts.

The **length** of a vector is the number of entries it has. (normally n)

Example. $\vec{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T.$

Vectors

We will be using vectors and matrices to store and manipulate data.

Definition: A **vector** $\vec{\mathbf{v}}$ is a column of numbers. Use bold faced letters or vector signs to distinguish vectors from other variables.

We refer to the **entries** of a vector by using subscripts.

The **length** of a vector is the number of entries it has. (normally n)

Example. $\vec{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$.

Example. Use a vector to represent the age distribution of a population: let F_i be the number of females with ages in the interval $\big[5i, 5(i+1)\big)$. We can represent the total female population by the vector $\vec{\mathbf{F}}$.
The females from 0 up to 5 are counted in F_0;
those from 5 up to 10 are counted in F_1, etc.

$$\vec{\mathbf{F}} = \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{n-1} \end{bmatrix}$$

Matrices

Definition: A **matrix** A is a two-dimensional array of numbers.

A matrix with m rows and n columns is called an $\overset{\text{“}m\text{ by }n\text{”}}{m \times n}$ matrix.

$\star$ Row by column — Row by column — Row by column $\star$

Note: A vector can be thought of as an $n \times 1$ matrix.

Matrices

Definition: A **matrix** A is a two-dimensional array of numbers.

A matrix with m rows and n columns is called an $\overset{\text{“}m \text{ by } n\text{”}}{m \times n}$ matrix.

⋆ Row by column — Row by column — Row by column ⋆

Note: A vector can be thought of as an $n \times 1$ matrix.

Matrices are denoted by a capital letter. Entries are lower case and have two subscripts, the corresponding row and column.

Example. A generic 2×3 matrix has the form $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$.

Matrices

Definition: A **matrix** A is a two-dimensional array of numbers.

A matrix with m rows and n columns is called an $\overset{\text{“}m\text{ by }n\text{”}}{m \times n}$ matrix.

⋆ Row by column — Row by column — Row by column ⋆

Note: A vector can be thought of as an $n \times 1$ matrix.

Matrices are denoted by a capital letter. Entries are lower case and have two subscripts, the corresponding row and column.

Example. A generic 2×3 matrix has the form $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$.

Definition: The matrix $B = \begin{bmatrix} 30 & 50 \\ 100 & 250 \end{bmatrix}$ is a **square matrix** because it has the same number of rows as columns.

Matrices

Example. We will sometimes interpret a matrix as a **transition** matrix.

In this case, the matrix is square (say $n \times n$), where the n rows and n columns correspond to certain **states** (situations).

An entry $a_{i,j}$ represents transitioning from state j to state i.

Matrices

Example. We will sometimes interpret a matrix as a **transition** matrix.

In this case, the matrix is square (say $n \times n$), where the n rows and n columns correspond to certain **states** (situations).

An entry $a_{i,j}$ represents transitioning from state j to state i.

Example. In our population example, suppose we want to model people getting older, transitioning from one state (age group) to the next. We would set up a transition matrix such as:

FROM state:

TO state:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

because everyone in the first age group will move to the second age group ($a_{2,1}$), everyone in state 2 will move to state 3 ($a_{3,2}$), etc.

Matrix Multiplication

The power of matrices arises in their multiplication.

Given two matrices, A of size $m \times k$ and B of size $l \times n$, we can find the product AB **if and only if** k equals l.

Let A be an $m \times k$ matrix and B, $k \times n$. Then AB is of size $m \times n$.

Matrix Multiplication

The power of matrices arises in their multiplication.

Given two matrices, A of size $m \times k$ and B of size $l \times n$, we can find the product AB **if and only if** k equals l.

Let A be an $m \times k$ matrix and B, $k \times n$. Then AB is of size $m \times n$.

To calculate the entries of AB, remember: "Row by column":

$$\begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 6 \\ -4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \bigcirc & \bigcirc & \bigcirc \\ \bigcirc & \bigcirc & \bigcirc \end{bmatrix}$$

Matrix Multiplication

The power of matrices arises in their multiplication.

Given two matrices, A of size $m \times k$ and B of size $l \times n$, we can find the product AB **if and only if** k equals l.

Let A be an $m \times k$ matrix and B, $k \times n$. Then AB is of size $m \times n$.

To calculate the entries of AB, remember: "Row by column":

$$\begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 6 \\ -4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \bigcirc & \bigcirc & \bigcirc \\ \bigcirc & \bigcirc & \bigcirc \end{bmatrix}$$

When we write A^2, this means AA; A^3 means AAA, etc.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \bigcirc & \bigcirc \\ 0 & 1 & \bigcirc \\ 0 & 0 & 1 \end{bmatrix}$$

The power of transition matrices

Example. Modeling a changing population using a matrix model.

Let us choose a size of age interval Δ=5 years ("Delta"), and divide the female population into states:

State 0: ages $[0, 5)$ with $F_0 = 150$ females
State 1: ages $[5, 10)$ with $F_1 = 200$ females
State 2: ages $[10, 15)$ with $F_2 = 180$ females
State 3: ages $[15, 20)$ with $F_3 = 120$ females
State 4: ages $[20, 25)$ with $F_4 = 60$ females

The power of transition matrices

Example. Modeling a changing population using a matrix model.

Let us choose a size of age interval Δ=5 years ("Delta"), and divide the female population into states:

age distribution vector:

State 0: ages $[0, 5)$ with $F_0 = 150$ females
State 1: ages $[5, 10)$ with $F_1 = 200$ females
State 2: ages $[10, 15)$ with $F_2 = 180$ females
State 3: ages $[15, 20)$ with $F_3 = 120$ females
State 4: ages $[20, 25)$ with $F_4 = 60$ females

$$\vec{\mathbf{F}} = \begin{bmatrix} 150 \\ 200 \\ 180 \\ 120 \\ 60 \end{bmatrix}$$

The power of transition matrices

Example. Modeling a changing population using a matrix model.

Let us choose a size of age interval Δ=5 years (“Delta”), and divide the female population into states:

age distribution vector:

State 0: ages $[0,5)$ with $F_0 = 150$ females
State 1: ages $[5,10)$ with $F_1 = 200$ females
State 2: ages $[10,15)$ with $F_2 = 180$ females
State 3: ages $[15,20)$ with $F_3 = 120$ females
State 4: ages $[20,25)$ with $F_4 = 60$ females

$$\vec{\mathbf{F}} = \begin{bmatrix} 150 \\ 200 \\ 180 \\ 120 \\ 60 \end{bmatrix}$$

Using a transition matrix, we can determine the population in 5 years:

$$A \cdot \vec{\mathbf{F}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^1 \begin{bmatrix} 150 \\ 200 \\ 180 \\ 120 \\ 60 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \\ 200 \\ 180 \\ 120 \end{bmatrix}$$

Leslie Matrices

The transition matrix in the previous example is not entirely realistic, because people die and are born

To take death into account, modify:

Leslie Matrices

The transition matrix in the previous example is not entirely realistic, because people die and are born

To take death into account, modify:

To take birth into account, modify:
(¡ females !)

Leslie Matrices

The transition matrix in the previous example is not entirely realistic, because people die and are born

To take death into account, modify:

To take birth into account, modify:
(¡ females !)

The resulting transition matrix is called a **Leslie matrix**:
Let m_i be the average number of females that women in state i bear.
Let p_i be the fraction of women in state i that survive to state $i+1$.

Leslie Matrices

The transition matrix in the previous example is not entirely realistic, because people die and are born

To take death into account, modify:

To take birth into account, modify:
(¡ females !)

The resulting transition matrix is called a **Leslie matrix**:
Let m_i be the average number of females that women in state i bear.
Let p_i be the fraction of women in state i that survive to state $i+1$.

$$\text{then} \begin{bmatrix} F_0(t+\Delta) \\ F_1(t+\Delta) \\ F_2(t+\Delta) \\ \vdots \\ F_{n-1}(t+\Delta) \end{bmatrix} = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} \\ p_0 & 0 & 0 & \cdots & 0 \\ 0 & p_1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & p_{n-2} & 0 \end{bmatrix} \begin{bmatrix} F_0(t) \\ F_1(t) \\ F_2(t) \\ \vdots \\ F_{n-1}(t) \end{bmatrix}$$

$$\vec{\mathbf{F}}(t+\Delta) = M \cdot \vec{\mathbf{F}}(t)$$

Leslie Matrices

Example. An animal population example (p. 47)
The population in three age groups, $F_0 = 80$, $F_1 = 40$, and $F_2 = 20$.

Suppose that as Δ time passes, everyone in state 2 dies, and one quarter of everyone else dies. Also suppose that the age-specific maternity rates are $m_0 = 0$, $m_1 = 1$, and $m_2 = 2$. Determine the Leslie matrix and the population distributions at times Δ and 2Δ.

$$\begin{bmatrix} & & \\ & 0 & 0 \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} 80 \\ 40 \\ 20 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \vec{\mathbf{F}}(\Delta)$$

$$\begin{bmatrix} & & \\ & 0 & 0 \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \vec{\mathbf{F}}(2\Delta)$$

Leslie Matrices

Example. Problem 1.5.6 from page 51.

(a) For the Leslie matrix $M = \begin{bmatrix} 3/2 & 2 \\ 1/2 & 0 \end{bmatrix}$, show that

$$M \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } M \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Leslie Matrices

Example. Problem 1.5.6 from page 51.

(a) For the Leslie matrix $M = \begin{bmatrix} 3/2 & 2 \\ 1/2 & 0 \end{bmatrix}$, show that

$$M \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } M \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(b) Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be any initial population. Find a and b so that

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = a \begin{bmatrix} 4 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Leslie Matrices

Example. Problem 1.5.6 from page 51.

(a) For the Leslie matrix $M = \begin{bmatrix} 3/2 & 2 \\ 1/2 & 0 \end{bmatrix}$, show that

$$M \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } M \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(b) Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be any initial population. Find a and b so that

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = a \begin{bmatrix} 4 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(c) Find $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ using parts (a) and (b).

Leslie Matrices

Example. Problem 1.5.6 from page 51.

(a) For the Leslie matrix $M = \begin{bmatrix} 3/2 & 2 \\ 1/2 & 0 \end{bmatrix}$, show that

$$M \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } M \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(b) Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be any initial population. Find a and b so that

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = a \begin{bmatrix} 4 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(c) Find $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ using parts (a) and (b).

(d) Show that the total population $P_n \approx P_0 2^n$.

Leslie Matrices

Example. Problem 1.5.6 from page 51.

(a) For the Leslie matrix $M = \begin{bmatrix} 3/2 & 2 \\ 1/2 & 0 \end{bmatrix}$, show that

$$M \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } M \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(b) Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be any initial population. Find a and b so that

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = a \begin{bmatrix} 4 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(c) Find $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ using parts (a) and (b).

(d) Show that the total population $P_n \approx P_0 2^n$.

- A Leslie matrix model is more descriptively realistic than the exponential model from Section 1.4, yet gives the same results.

Leslie Matrices

Example. Problem 1.5.6 from page 51.

(a) For the Leslie matrix $M = \begin{bmatrix} 3/2 & 2 \\ 1/2 & 0 \end{bmatrix}$, show that

$$M \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } M \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(b) Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be any initial population. Find a and b so that

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = a \begin{bmatrix} 4 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(c) Find $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ using parts (a) and (b).

(d) Show that the total population $P_n \approx P_0 2^n$.

- A Leslie matrix model is more descriptively realistic than the exponential model from Section 1.4, yet gives the same results.
- We've just worked with eigenvalues and eigenvectors!

Markov Chains

A **Markov chain** is a sequence of random variables from some sample space, each corresponding to a successive time interval. From one time interval to the next, there is a *fixed* probability $a_{i,j}$ of transitioning from state j to state i. No transition depends on a past transition.

Keep track of these probabilities in an associated transition matrix A.

Markov Chains

A **Markov chain** is a sequence of random variables from some sample space, each corresponding to a successive time interval. From one time interval to the next, there is a *fixed* probability $a_{i,j}$ of transitioning from state j to state i. No transition depends on a past transition.

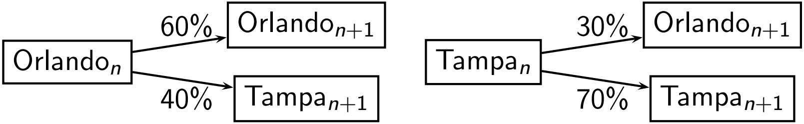
Keep track of these probabilities in an associated transition matrix A.

Markov Chains

A **Markov chain** is a sequence of random variables from some sample space, each corresponding to a successive time interval. From one time interval to the next, there is a *fixed* probability $a_{i,j}$ of transitioning from state j to state i. No transition depends on a past transition.

Keep track of these probabilities in an associated transition matrix A.

Example. Suppose you run a rental company based in Orlando and Tampa, Florida. People often drive between the cities; cars can be picked up and dropped off in either city. Suppose that historically,



What distribution of cars can the company expect in the long run?

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

FROM:

$$\textbf{TO}: \begin{array}{c} \\ \text{Or} \\ \text{Tm} \end{array} \begin{array}{c} \begin{array}{cc} \text{Or} & \text{Tm} \end{array} \\ \begin{bmatrix} 0.6 & \\ & \end{bmatrix} \end{array} = A,$$

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

FROM:

TO:	Or	Tm
Or	0.6	0.3
Tm	0.4	0.7

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} = A,$$

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

FROM: Or, Tm; **TO**: Or, Tm

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} = A,$$

- Let o_n be the probability that a car is in Orlando on day n
- Let t_n be the probability that a car is in Tampa on day n.

We can represent the distribution of cars at time n with the vector

$$\vec{\mathbf{x}}_n = \begin{bmatrix} o_n \\ t_n \end{bmatrix}.$$

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

FROM:

$$\textbf{TO}: \begin{matrix} & \text{Or} & \text{Tm} \\ \text{Or} & 0.6 & 0.3 \\ \text{Tm} & 0.4 & 0.7 \end{matrix} \quad \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} = A,$$

- Let o_n be the probability that a car is in Orlando on day n
- Let t_n be the probability that a car is in Tampa on day n.

We can represent the distribution of cars at time n with the vector $\vec{\mathbf{x}}_n = \begin{bmatrix} o_n \\ t_n \end{bmatrix}$. And so, $\vec{\mathbf{x}}_{n+1} = \begin{bmatrix} o_{n+1} \\ t_{n+1} \end{bmatrix} = A \cdot \begin{bmatrix} o_n \\ t_n \end{bmatrix} = A\vec{\mathbf{x}}_n$.

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

FROM:

$$\textbf{TO}: \begin{matrix} & \text{Or} & \text{Tm} \\ \text{Or} & \multirow{2}{*}{} \\ \text{Tm} \end{matrix} \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} = A,$$

- Let o_n be the probability that a car is in Orlando on day n
- Let t_n be the probability that a car is in Tampa on day n.

We can represent the distribution of cars at time n with the vector $\vec{\mathbf{x}}_n = \begin{bmatrix} o_n \\ t_n \end{bmatrix}$. And so, $\vec{\mathbf{x}}_{n+1} = \begin{bmatrix} o_{n+1} \\ t_{n+1} \end{bmatrix} = A \cdot \begin{bmatrix} o_n \\ t_n \end{bmatrix} = A\vec{\mathbf{x}}_n$.

Given an initial distribution $\vec{\mathbf{x}}_0 = \begin{bmatrix} o_0 \\ t_0 \end{bmatrix}$,

the expected distribution of cars at time n is $\vec{\mathbf{x}}_n = A^n\vec{\mathbf{x}}_0$.

Markov Chains

For example, if they company starts off with twice as many cars in Orlando as in Tampa, then $\vec{\mathbf{x}}_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$, so we expect

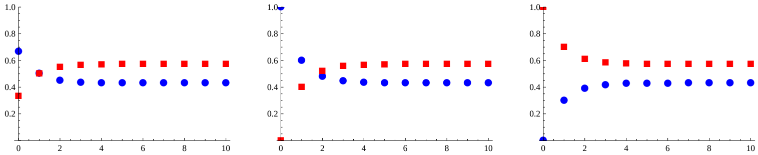
$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

Markov Chains

For example, if they company starts off with twice as many cars in Orlando as in Tampa, then $\vec{\mathbf{x}}_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$, so we expect

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \qquad \\ \qquad \end{bmatrix}.$$

$$\vec{\mathbf{x}}_2 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} \qquad \\ \qquad \end{bmatrix} = \begin{bmatrix} \qquad \\ \qquad \end{bmatrix}.$$

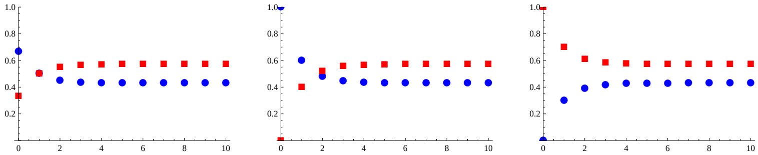


Markov Chains

For example, if they company starts off with twice as many cars in Orlando as in Tampa, then $\vec{\mathbf{x}}_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$, so we expect

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

$$\vec{\mathbf{x}}_2 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$



How do we determine the expected distribution in the long run?

Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.

Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.
[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.
[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

In our example, the equilibrium distribution satisfies

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix} = \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix}.$$

Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.

[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

In our example, the equilibrium distribution satisfies

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix} = \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix}.$$

So solve: $0.6o_{eq} + 0.3t_{eq} = o_{eq}$ and $0.4o_{eq} + 0.7t_{eq} = t_{eq}$.
Both equations reduce to $0.3t_{eq} = 0.4o_{eq}$, so $o_{eq} = \frac{3}{4}t_{eq}$.

Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.
[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

In our example, the equilibrium distribution satisfies

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix} = \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix}.$$

So solve: $0.6o_{eq} + 0.3t_{eq} = o_{eq}$ and $0.4o_{eq} + 0.7t_{eq} = t_{eq}$.
Both equations reduce to $0.3t_{eq} = 0.4o_{eq}$, so $o_{eq} = \frac{3}{4}t_{eq}$.

Conclusion: If the company has 7000 cars in all, they would expect that in the long run, ____________________

Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.
[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

In our example, the equilibrium distribution satisfies

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix} = \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix}.$$

So solve: $0.6o_{eq} + 0.3t_{eq} = o_{eq}$ and $0.4o_{eq} + 0.7t_{eq} = t_{eq}$.
Both equations reduce to $0.3t_{eq} = 0.4o_{eq}$, so $o_{eq} = \frac{3}{4}t_{eq}$.

Conclusion: If the company has 7000 cars in all, they would expect that in the long run, ________________________________

In Markov Chains: ★ The sum of the entries in every column of A is 1, because the total probability of transitioning **from** state i is 1.

Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.
[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

In our example, the equilibrium distribution satisfies

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix} = \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix}.$$

So solve: $0.6o_{eq} + 0.3t_{eq} = o_{eq}$ and $0.4o_{eq} + 0.7t_{eq} = t_{eq}$.
Both equations reduce to $0.3t_{eq} = 0.4o_{eq}$, so $o_{eq} = \frac{3}{4}t_{eq}$.

Conclusion: If the company has 7000 cars in all, they would expect that in the long run, ________________________________

In Markov Chains: ★ The sum of the entries in every column of A is 1, because the total probability of transitioning **from** state i is 1.

★ There is no general rule for what the row sum will be.