Course Notes

Mathematical Models, Spring 2013

Queens College, Math 245

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Then we must **analyze our models** to determine their applicability.

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- What is happening? (Observation)
- What are the reasons for the behavior? (Hypothesis)
- How do we convey that our reasoning is plausible? ("proof")

— Use the language of mathematics! —

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- Describe mathematically. Assign each quantity a variable. Represent each relationship with an equation.

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  And *proportional* means $v = ax$ for some constant a. (Goal?)

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Something is not quite right...

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- Start the modeling process over.

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Perhaps the proportionality assumption is incorrect?

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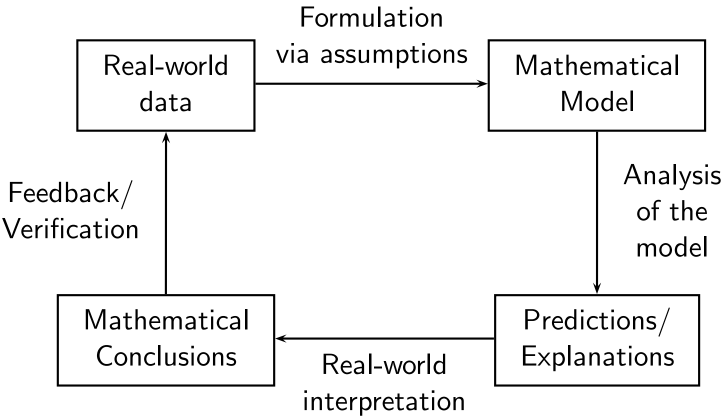
(Although not all!)

The Modeling Process

This chart summarizes the modeling process.



To do well in this class:

- **Come to class prepared.**
  - Print out and read over course notes.
  - Read assigned sections before class.
- **Form good study groups.**
  - Discuss homework and classwork.
  - Final project is a group project.
  - You will depend on this group.
- **Put in the time.**
  - Three credits = (at least) nine hours / week out of class.
  - Homework stresses key concepts from class; learning takes time.
- **Stay in contact.**
  - If you are confused, ask questions (in class and out).
  - Don't fall behind in coursework or project.
  - I need to understand your concerns.

All homeworks posted online; first one due Monday.