# Course Notes

## Mathematical Models, Spring 2013

Queens College, Math 245

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# What is a model?

A model is an object or concept used to represent something else.
It converts reality to a form we can comprehend.

- Reality: How to understand the aerodynamics of an airplane?
  Model: Use a **model airplane** or a **computer simulation**.
- Reality: Politics flows between left-wing and right-wing ideas.
  Model: Think of public opinion as a **pendulum**.

A mathematical model is a model involving mathematical concepts.

**IN THIS CLASS:**
We take real-world situations and represent them using mathematics.

- Model the position of a falling object by **function fitting**.
- Model people waiting using a **computer simulation**.
- Model allocating resources using a **system of inequalities**.

Then we must **analyze our models** to determine their applicability.

# **Why** should we model?

As scientists, we want to understand how the world works.

$\rightarrow$ $\rightsquigarrow$ $\rightarrow$ observed real-world behavior

- What is happening? (Observation)
- What are the reasons for the behavior? (Hypothesis)
- How do we convey that our reasoning is plausible? ("proof")

— Use the language of mathematics! —

# Steps of the Modeling Process

**Goal:** Understand what is involved in “mathematical modeling”.

**First Step: Formulation.**

- State the question. If the question is vague, make it precise. If the question is too big, subdivide it into manageable parts.
- Identify factors. Decide which quantities influence the behavior. Determine relationships between the quantities.

  [**Important:** *we are introducing* ______________________.]
- Describe mathematically. Assign each quantity a variable. Represent each relationship with an equation.

# Motivating Example: Gravity by Galileo

**Example.** In Galileo's time, a key question changed from:

*Why* do objects fall? –to– *How* do objects fall?
(Philosophical question) (Describe a falling object's velocity)

**First Step: Formulation.**

- State the question. (Precise! What is an answer?)
  What formula describes an object's position as it falls?
- Identify factors. Galileo chose only distance, time, and velocity.
  Other variables?: ____________________

Simplifying Assumption: Velocity is proportional to the distance fallen.

- Describe mathematically.
  Assign variables. Call distance $x$, time $t$, and velocity $v$.
  Then relationships give equations:
  Velocity and distance are related: $v = \frac{dx}{dt}$.
  And *proportional* means $v = ax$ for some constant $a$. (Goal?)

# Steps of the Modeling Process

- After the formulation step, we have variables and equations.
- Do some analysis to develop **mathematical conclusions**.

**Second Step: Mathematical Manipulation.**

This may entail one or more of:

- Calculations
- Solving an equation
- Proving a theorem
- Other...

*In our gravity example,*

We have both $v = \frac{dx}{dt}$ and $v = ax$. Set them equal.

This gives the (differential) equation: $\frac{dx}{dt} = ax$.

Solving gives that $x(t) = ke^{at}$ for constants $a$ and $k$.

Something is not quite right...

# Steps of the Modeling Process

We have a mathematical conclusion, but does it give a "right answer"?

The *most important* step of the modeling process is:

**Third Step: Evaluation.**
Translate the results back to the real-world situation and ask questions:

- Has the model explained the real-world observations?
- Are the answers we found accurate enough?
- Were our assumptions good assumptions?
- What are the strengths and weaknesses of our model?
- Did we make any mistakes in our mathematical manipulations?

If there are any problems,

- **Go back** to the First Step, Formulation.
- Change your assumptions!
- Start the modeling process over.

# Motivating Example: Gravity by Galileo

**Third Step: Evaluation.**

Our mathematical calculations imply that the position of a falling object is $x(t) = ke^{at}$.

In our real-world situation, we can set initial position to be 0. Mathematically, $x(0) = 0$.

This lets us solve for $k$ in our equation:
$0 = x(0) = ke^{a0} = ke^{0} = k$.

So $k = 0$. Plugging into our equation implies $x(t) = 0$.

In words, this means that our object stays at rest for all $t$.

EPIC FAIL!

Perhaps the proportionality assumption is incorrect?

# Motivating Example: Gravity by Galileo

**First Step: Formulation.**

Alternate assumption: The velocity is proportional to the time it has been falling. In particular, the velocity increases by 32 ft/sec.

Mathematically, we have the equations $v = 32t$ and $v = \frac{dx}{dt}$.

**Second Step: Mathematical Manipulation.**

Integrating gives $x(t) = 16t^2 + C$.

Since $x(0) = 0$ we can find $C = 0$.

Therefore an object falling from rest has position $x(t) = 16t^2$.

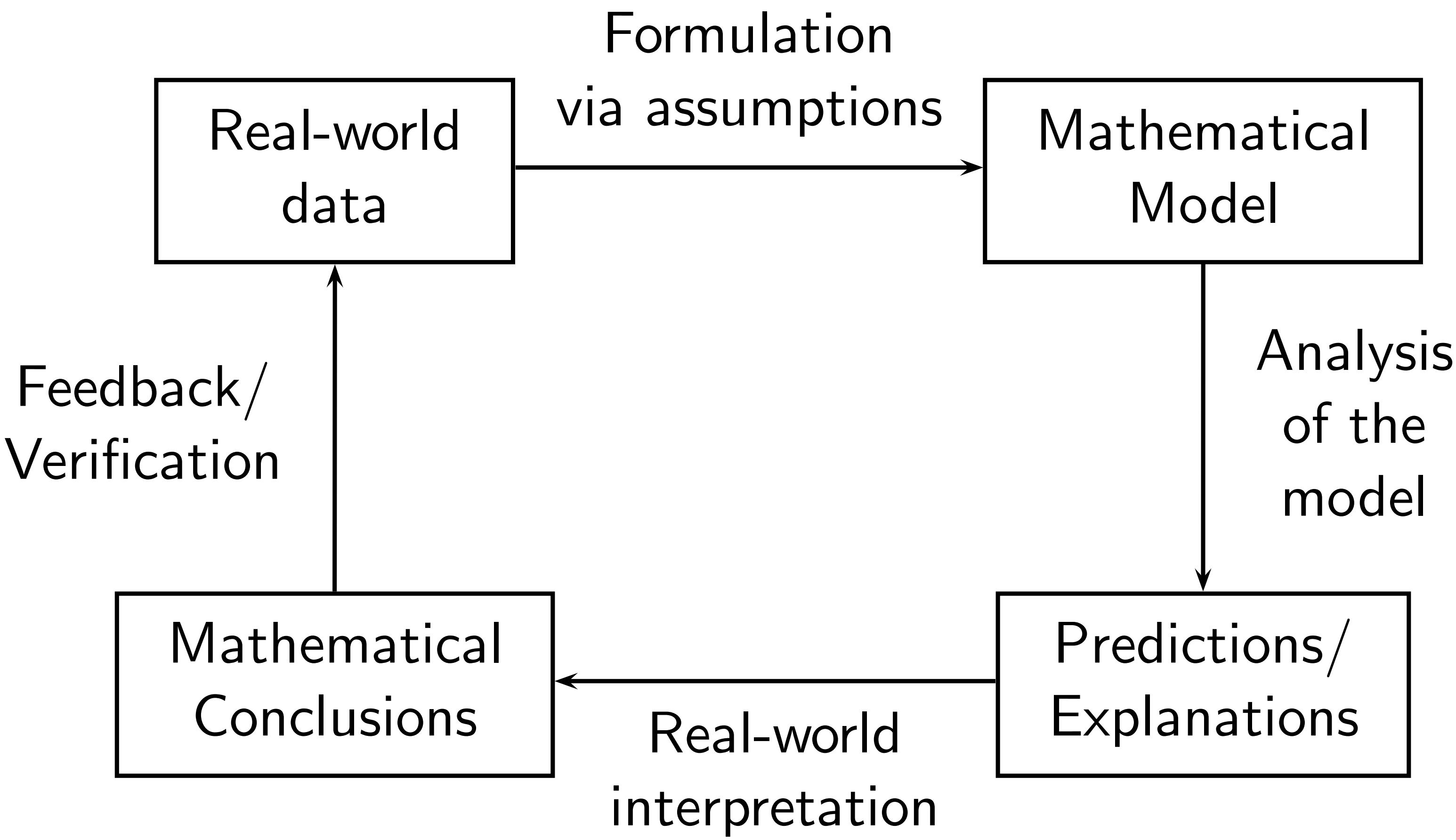
**Third Step: Evaluation.**

This function agrees well with observations in many instances.

*(Although not all!)*

# The Modeling Process

This chart summarizes the modeling process.



# To do well in this class:

- **Come to class prepared.**
  - Print out and read over course notes.
  - Read assigned sections before class.
- **Form good study groups.**
  - Discuss homework and classwork.
  - Final project is a group project.
  - You will depend on this group.
- **Put in the time.**
  - Three credits = (at least) nine hours / week out of class.
  - Homework stresses key concepts from class; learning takes time.
- **Stay in contact.**
  - If you are confused, ask questions (in class and out).
  - Don't fall behind in coursework or project.
  - I need to understand your concerns.

All homeworks posted online; first one due Monday.