Linear Optimization

Today we start our last topic of the semester, linear optimization.

Comprehension goals:

- What is a linear program?
- Visualizing linear programs graphically.
- Understanding solutions graphically.
- Solving linear programs using *Mathematica*

Fertilizer example (p.253)

(p.253)

A fertilizer manufacturer uses nitrates and phosphates to make batches of two different kinds of fertilizer.

- Sod-King fertilizer needs 4 phosphates, 18 nitrates.
- Gro-Turf fertilizer needs 1 phosphate, 15 nitrates.

The profit for one batch of Sod-King is $1000.
The profit for one batch of Gro-Turf is $500.

The company has 10 phosphates and 66 nitrates on hand.

Question. How many batches of each should the company make to earn the most profit?

Initial thoughts?

Fertilizer example (p.253)

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Translate the problem into mathematics:
We must determine how many batches to make of each.

- Let x represent the number of batches of Sod-King made.
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What are the constraints on what x and y can be?

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What are we trying to maximize?

- Profit:

Linear Programs

Maximize $1000x + 500y$

$$\text{subject to the constraints:}\quad \begin{aligned} 4x + y &\le 10 \\ 18x + 15y &\le 66 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

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&\text{subject to the constraints:} \quad
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4x + y & \leq & 10 \\
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\end{aligned}$$

This is a **linear program**, an optimization problem of the form:

$$\begin{aligned}
&\text{Maximize } c_1x_1 + c_2x_2 + \cdots + c_nx_n \text{ (the \textbf{objective function})} \\
&\text{subject to (the \textbf{constraints}):} \quad
\begin{array}{rcl}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \leq & b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \leq & b_2 \\
\vdots & & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \leq & b_m
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- The decision variables can have any real value, not only integers.
- All constraints and the objective functions are *linear combinations* of the decision variables. (Coefficients are constants.)
- A linear program in the above form is "easy to solve".

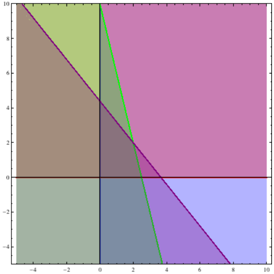
Fertilizer example, graphically

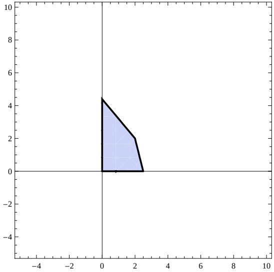
Maximize $1000x + 500y$

subject to the constraints:

$$\begin{aligned} 4x + y &\leq 10 \\ 18x + 15y &\leq 66 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Let's consider our example graphically.





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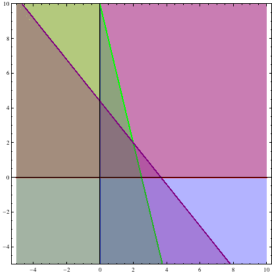

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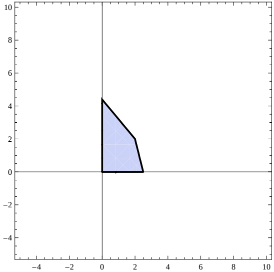

Definition: The set of points (x, y) that satisfy the constraints is called the **feasible region**.

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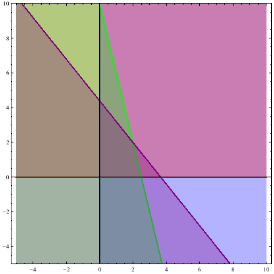

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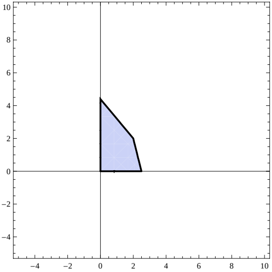
- In general, points of form $(x_1, x_2, \ldots, x_n)$.
- Feasible region always a polytope. (Always has flat sides and is convex.)
- Feasible region may be bounded or unbounded; might be empty.



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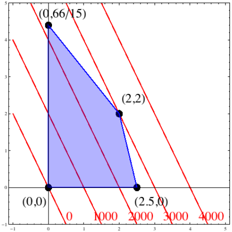

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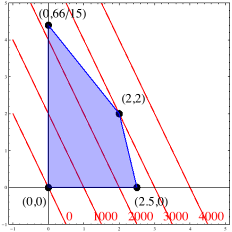

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Is there a point in the feasible region such that $1000x + 500y = 2000$?

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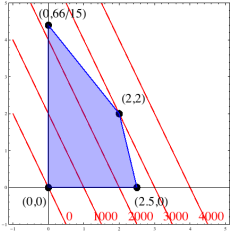

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Is there a point in the feasible region such that $1000x + 500y = 2000$?
Is there a point in the feasible region such that $1000x + 500y = 4000$?

As we plot these **constant-objective** lines, we notice that

- They are parallel.

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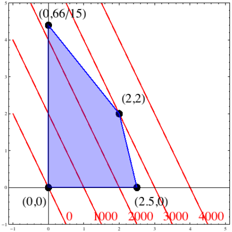

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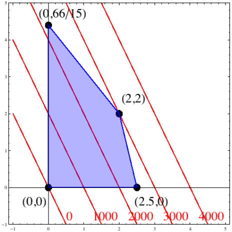

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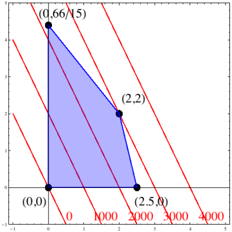

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- They are parallel.
- If there is a feasible region, at least one line will intersect it.
- As we increase the "constant", the last place we touch the feasible region is on the boundary, at one or more corners.

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We have intuited the following theorem.

Theorem. The maximum (or minimum) in a linear program either:

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4. Pick out the optimum value.

Solution of fertilizer example

Maximize $p(x,y) = 1000x + 500y$

subject to the constraints:

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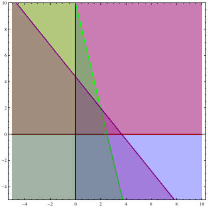

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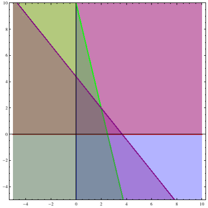

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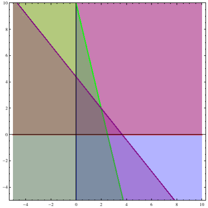

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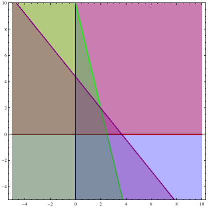

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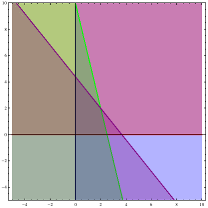

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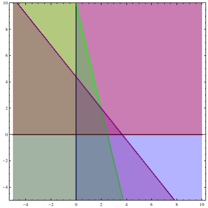

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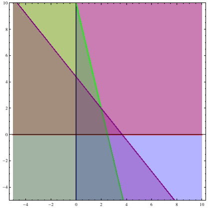

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3. Evaluate the objective function at each corner point.
   - $p(0,0) = 0$
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   - $p(2,2) = 3000$.

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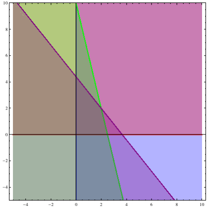

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4. Pick out the optimum value. [Max value: $3000, occurs at (2,2).]

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```
In[1]: Maximize[{1000 x + 500 y,
         x >= 0 && y >= 0 && 4 x + y <= 10 && 18 x + 15 y <= 66}, {x,y}]
Out[1]: {3000, {x -> 2, y -> 2}}
```

Using *Mathematica* to solve a linear program

Once you have written your optimization problem as a linear program, you can use *Mathematica* to solve your problem.

Use either the `Maximize` or `Minimize` command.

Syntax: `Maximize[{`*obj*`,`*constr*`},`*vars*`]`

- *obj* is the objective function that you wish to optimize.
- *constr* are the set of all constraints, joined with `&&`'s (ANDs).
- *vars* is the set of variables.

```
In[1]: Maximize[{1000 x + 500 y,
         x >= 0 && y >= 0 && 4 x + y <= 10 && 18 x + 15 y <= 66}, {x,y}]
Out[1]: {3000, {x -> 2, y -> 2}}
```

The output gives the optimum value and the values the variables take on there.