Generalizations of Polar Coordinates

When 2-dim'l regions D have radial flavors, we use polar coordinates.

When 3-dim'l regions E have radial flavors, there are two choices:

Cylindrical coordinates

A point can have coords (r, θ, z):

(r, θ) are polar coords of xy-plane
(r is the distance from the z-axis)
and z is the "distance" to xy-plane

Useful when problems involve *symmetry about an axis*. Cylinder, Paraboloids, Cones w/flat bases

Spherical coordinates

A point can have coords (ρ, θ, ϕ).

ρ is the distance from the origin
θ is same as in polar
ϕ is $\angle$ between $+z$ axis and $\overline{OP}$.

Useful when problems involve *symmetry about a point*
Spheres, Cones with curved bases.

Converting from cartesian coordinates

Cylindrical coordinates

Conversion:

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$

$$dV = r\,dr\,d\theta\,dz$$

Spherical coordinates

Conversion:

$x = \rho\sin\phi\cos\theta$

$z = \rho\cos\phi \qquad y = \rho\sin\phi\,\sin\theta$

$\rho^2 = x^2 + y^2 + z^2; \qquad \tan\phi = \frac{\sqrt{x^2+y^2}}{z}$

$$dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

Practice

Cylindrical coordinates

Practice changing coordinates:

$(r,\theta,z)=(2,\frac{2\pi}{3},1)$; $(x,y,z)=(3,-3,7)$

Identify cyl. coord. equations:

2. $r=2$; $z=r^2$; $r^2-2z^2=4$

3. Sketch $r^2\le z\le 2-r^2$

Convert to cylindrical coordinates

4. $x^2+y^2+z^2=2$; $x^2+y^2=2y$

5. Give solid between $x^2+y^2=1$ and $x^2+y^2+z^2=4$.

6. $$\left\{\begin{array}{c} -2\le x\le 2 \\ -\sqrt{4-x^2}\le y\le\sqrt{4-x^2} \\ \sqrt{x^2+y^2}\le z\le 2 \end{array}\right\}$$

Spherical coordinates

Practice changing coordinates:

$(\rho,\theta,\phi)=(2,\frac{\pi}{4},\frac{\pi}{3})$;$(x,y,z)=(-1,1,\sqrt{6})$

Identify sph. coord. equations:

2. $\phi=\frac{\pi}{3}$; $\rho\sin\phi=2$; $\rho=2\cos\phi$

3. Sketch $\left(2\le\rho\le 3 \ \&\ \frac{\pi}{2}\le\phi\le\pi\right)$

Sketch $\left(0\le\phi\le\frac{\pi}{3} \ \&\ \rho\le 2\right)$

Convert to spherical coordinates

4. $z=x^2+y^2$; $z=x^2-y^2$

5. Give solid inside $x^2+y^2+z^2=4$, above xy-plane, below $z=\sqrt{x^2+y^2}$.

6. $$\left\{\begin{array}{c} 0\le x\le 1 \\ 0\le y\le\sqrt{1-x^2} \\ \sqrt{x^2+y^2}\le z\le\sqrt{2-x^2-y^2} \end{array}\right\}$$