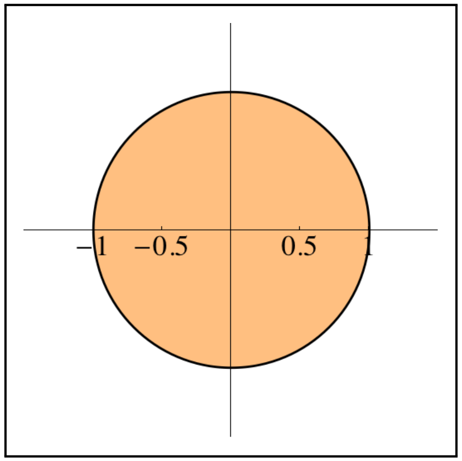
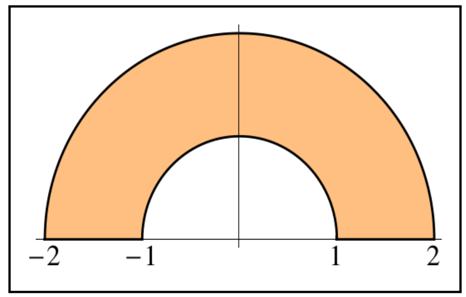
Some regions demand polar coordinates

Some regions are best described in polar coordinates:



Parts of circles



Parts of annuli

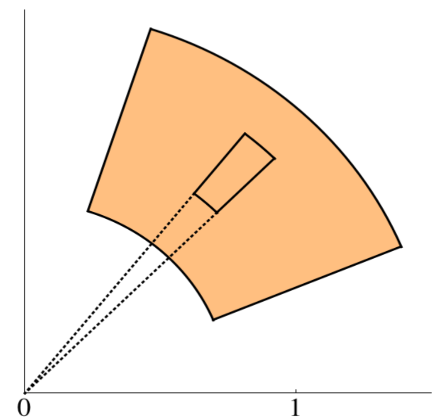
These are called **polar rectangles** because they have the form

$$\left\{ \begin{array}{c} a \leq r \leq b \\ \alpha \leq \theta \leq \beta \end{array} \right\} \text{ for constants } \left\{ \begin{array}{c} a,\ b \\ \alpha,\ \beta \end{array} \right\}$$

Polar coordinates

Goal: Convert a double integral involving x's and y's
into a double integral involving r's and θ's.

Important: Using "polar slices"
introduces a complication.

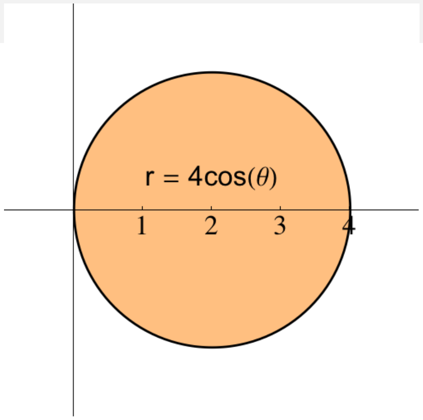


In this picture, dA is **not** $dr\,d\theta$.
The radial component is ____ and the circular component is _____.
This means $dA =$ __________. (How to remember?)

Consequence: To calculate $\iint_D f(x, y)\,dA$,
when D is best described in polar coordinates, calculate

$$\iint_D f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Polar coordinate example



Example. Find the area inside $r = 4\cos\theta$ from $\theta = \frac{\pi}{4}$ to $\frac{\pi}{2}$.

This region is defined as $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$ and ________ $\leq r \leq$ ________.

We use $A = \iint dA$:

★ Along the way, we had $\displaystyle\int_{\alpha}^{\beta} \frac{r^2}{2}\, d\theta$........

Application: Density (p. 692/723)

Example. The density of a point on a semicircular lamina of radius a is proportional to the distance from the center of the circle. Find the mass of the lamina.

Solution. **Draw a picture!**

Setup: Let the lamina be the upper half of the circle $x^2 + y^2 = a^2$, which is a polar rectangle:

The density function can be written:
$\rho(x, y) = K\sqrt{x^2 + y^2}$

The total mass is $m = \iint_D \rho(x, y)\, dA$

$$\cdots = \int_{\theta=0}^{\theta=\pi} K\frac{a^3}{3}\, d\theta = \left[K\frac{a^3}{3}\theta\right]_{\theta=0}^{\theta=\pi} = \frac{\pi K a^3}{3}.$$

Changing from (x, y) to (r, θ)

Example. Calculate $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\, dy\, dx$ $\left(\begin{array}{c}\text{hint for}\\\text{polar!}\end{array}\right)$

Solution. **Draw a picture!**

Notice that this is the polar rectangle
$0 \le r \le 3$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Rewrite the integral as

$$\iint_D (r^3 \cos^3\theta + r^3 \cos\theta \sin^2\theta)\, r\, dr\, d\theta$$

$$= \iint_D r^4 \cos\theta(\cos^2\theta + \sin^2\theta)\, dr\, d\theta = \iint_D r^4 \cos\theta\, dr\, d\theta$$

$$= \left(\int_{r=0}^{r=3} r^4\, dr\right) \cdot \left(\int_{\theta=-\pi/2}^{\theta=\pi/2} \cos\theta\, d\theta\right)$$

$$= \left[\frac{r^5}{5}\right]_{r=0}^{r=3} \cdot \left[\sin\theta\right]_{\theta=-\pi/2}^{\theta=\pi/2} = \left(\frac{3^5}{5} - 0\right) \cdot (1 - (-1)) = \frac{2 \cdot 3^5}{5}$$

Volume example

Example. Find the volume of the solid under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $(x-1)^2 + y^2 = 1$.

Plan of attack: **Draw a picture!**

- Notice circley thingees. Think: Possible Polar Problem.
- Convert the given information to polar equations using

$$(x, y) = (r\cos\theta, r\sin\theta):$$

$$(x-1)^2 + y^2 = 1 \rightsquigarrow (r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1 \rightsquigarrow r = 2\cos\theta.$$

$$(x^2 + y^2) \rightsquigarrow r^2.$$

- Set up the polar integral.

  So $\iint_D (x^2 + y^2)\, dA = \iint_D r^2\, r\, dr\, d\theta$.

- Integrate...