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This problem must have an absolute maximum, which must occur at a critical point. (Why?) Therefore $(x, y, z) = (2, 2, 1)$ is the absolute maximum, and the maximum volume is $xyz = 4$.

Optimization subject to constraints

The method of Lagrange multipliers is an alternative way to find maxima and minima of a function $f(x, y, z)$ subject to a given constraint $g(x, y, z) = k$.

Motivating Example. Suppose you are trying to find the maximum and minimum value of $f(x, y) = y - x$ when we only consider points on the curve $g(x, y) = x^2 + 4y^2 = 36$.

What should we do?

The method of Lagrange multipliers

To find the maxima and minima of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (as long as $\nabla g \neq 0$ on this constraint)

- Solve for all tuples (x, y, z, λ) such that

$$\nabla f(x, y, z) = \lambda \cdot \nabla g(x, y, z) \quad \textbf{and} \quad g(x, y, z) = k$$

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  - The largest f value corresponds to a maximum
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- Careful about when this applies. $(\nabla g \neq 0)$

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Goal: Maximize $V = xyz$ subject to $g(x, y, z) = 2xz + 2yz + xy = 12$.

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By the method of Lagrange multipliers, we need to solve:

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$$\text{Solve: } \begin{cases} yz = \lambda(2z + y) \\ xz = \lambda(2z + x) \\ xy = \lambda(2x + 2y) \\ 2xz + 2yz + xy = 12 \end{cases}$$

- Four equations, four unknowns, so possibly solvable.

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- Eliminate λ using first two equations. (& that $\lambda \neq 0$ by Eq. (4).)
- Multiply Eq. (2) by y, Eq. (3) by z, simplify.

Why does this work?

For functions of two variables:

The tangent line to the level curve $g(x, y) = k$ and the level curve $f(x, y) =$max are parallel, so their normals are too. We conclude that $\nabla f(x, y) = \lambda g(x, y)$.

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For functions of three variables:

The tangent plane to the level curve $g(x, y, z) = k$ and the level curve $f(x, y, z) =$max are parallel, so their normals are too. We conclude that $\nabla f(x, y, z) = \lambda g(x, y, z)$.

Another example

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- Check for critical points on the interior of the region.
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Solution?